A Numerical Approach to Profile Investor Preferences from Option Prices

Master Thesis submitted to

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Abstract

This paper introduces a behavioural model and an algorithm that allow define classes of investors and draw the size each of them from financial data. The nonparametric pricing kernel estimated from stocks and options quotes allows to derive an estimate of the market utility. At the micro level it is assumed that each individual perceives lower future returns and higher future returns differently, and at a given threshold the individual switches from one attitude to the other. This switching point is peculiar of each class.
The aggregate of the individual utilities must have the same features of the estimated market utility function, and two random search algorithms to compute the optimal aggregation are proposed and compared. Both computation techniques provide a similar distribution of investor classes. When the markets are bearish, even negative future states are perceived as high. On the other hand when markets are performing well, the switch to the high perception occurs only for bigger returns. For stock markets without a clear trend there is no predominance of a single class, investors are split between '"early"' and '"late"' switchers.

Keywords:
pricing kernel, power utility, behavioural finance, numerical solution

Dedication

The topic this paper deals with is extremely complex and interesting. I first want to thank Prof. Härdle and R. Moro for the several opportunity to learn from them and the support during the production of this paper.
Getting familiar with so many different concepts and bringing together in a choherent form has been an extraordinary exercise. To overcome all these challenges, the moral support of several friends and of my family has been very precious to me. I therefore want to thank Mario Beppato, Lipi Banerjee, Stefano Castruccio, Stefano Mattia, as well as Cristina, Massimo and Michela Marzetti for their great presence.

You have to look at trades to the lens of your portfolio's utility if you want to understand if they are truly adding value

Kenenth Griffin, Founder and CEO of Citadel Investment
Risk 20, July 2007, Page 36.

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Chapter 1

Introduction

A good knowledge of the sentiment of the market is crucial in several professions related to financial markets, as for instance asset and risk management. The work presentet here does not just aim at describing the German capital markets as an aggregate, but it also offers insights in the behaviour of the individual investors, taking into account the irrational elements that affect their decision making process. There are two peculiar features that distinguish the theory presented here from the standard economic models. Instead of using a representative agent the market is modelled here as a compound of investors, where market prices are obtained from the aggregation of the prices expressed by each individual. Furthermore each investor type that belongs to the market has a peculiar utility function that reflects its different expectations about future states. As a result it is eventually possible to define profiles of investors and size each of them is, making therefore possible to identify market trends and spot niches.

The theory presented in the following chapters has been initially assembled by Moro, Detlefsen, and Härdle (2007). They start from a simple market model and derive the market empirical pricing kernel and eventually the market utility function. Once the preferences of the representative aggregate investor are estimated, models for the individual investors utility can be tested to verify whether they are consistent with the market data.

The hypothesis brought forward is that investors have a kinked utility function, shaped like two segments of two curves that intersect in a given point. The two segments belong to two utility functions for two different utility levels. One utility level incorporates the preferences of an investor who is perceiving a given return as low. The set of preferences of an investor that undertakes a low attitude is assumed to be the same over all the investor types. The second utilty level incorporates the preferences of investors that are perceiving a given return as high. The perception of high returns is dif-

ferent for each and every investor type. As a conclusion, the switching point between the two segments is charachteristic of each investor class. Each profile of investors switches from the defensive to the aggressive attitude in a profile-specific state. For each investor type, all the states associated with returns below the switching point will be perceived according to the low attitude, and all the states associated with higher returns than the switching point will be perceived as high.
Once the types have been defined, the aggregation problem has to be solved in order to attribute to each class a weight. The aggregation is in fact performed assuming that the return associated with a level of the market utility function is equivalent to the weighted average of the returns that each investor type associate with that level of utility. This approach leads to an ill-posed aggregation problem that can't be solved analytically. Two numerical procedures are then introduced to estimate the optimal allocation of weights. Several tests are performed to show how the final result is sensitive to the parameter choice. Under certain parameter choices both procedures deliver similar, smooth results and good fit to the market data.
A potential useful application can be found in asset management. From the asset manager perspective it is known that the marketability of a financial product not only depends on the return provided, but also on the risk profile of the product and on how the risk is perceived by investors. Simply put, a financial product that promises high yield is not worth much if the market believes that this return can hardly be achieved. However risk, when synonym of volatility, is a double face concept that has to do both with the prudence and with the aspirations of the investors. From one side it quantifies the danger of incurring into losses, but at the same time it represents the opportunity to achieve an higher return. The aggregation theory, proposed by Moro et al. (2007) and implemented in this paper, allows to describe how the investors perceive future returns and riskiness.
It is in fact observed that when the market has a clear growing trend, the investors switch to the high attitude for positive returns. On the other hand when the market has a clear negative trend the investors undertake the high attitude even for negative returns.
The switching point can be seen as the point where future states are perceived in a fair way. The perception of other returns are affected by a number of other consideration that individuals do concerning risk, initial endowment, expectations and other irrational element. Therefore the switching point can be also interpreted as the benchmark against which performances are compared by each investor class. Asset managers can fine tune their product range to better meet the preference sets of all the different investor types.
The foundation upon which Moro, Detlefsen, and Härdle (2007) have built

their model of the market is the asset pricing book of Cochrane (2001). The model presented by Cochrane (2001) is a standard didactic tool that combines simplicity with an effective representation of several properties of capital markets, under the traditional outlook of the representative agent theory.

Another big contribution to the model presented here is the research from Jackwerth (2002) about the interrelations between empirical pricing kernels and risk aversion. Furthermore Brown and Jackwerth (2005) collects in a comprehensive overview all the attempts made to explain the pricing kernel puzzle. While Jackwerth (2002) brightly present the technique for deriving the market risk aversion from option prices, Brown and Jackwerth (2005) propose a model of utility based on two levels similar to the one described further in the third chapter.

The economic utility is a measure that identifies how an agent qualifies a given state of the world, assigning to it a score. Several experiments have been attempted over time to measure how preferences are formed. In asset pricing it is well known that a security that pays a given amount of money in one year has higher present value than a security that pays the same amount of money after a larger period. In the same way it is known that a certain payoff has an higher present value than a security with identical expected value but more unstable payoffs. Discount factors for time and risk should therefore be carefully used to describe properly the preferences of individuals.

Nevertheless the research from Kahneman and Tversky (1979) proves empirically that individuals are risk averse when comparing prospects that have positive payoffs, but become risk seeking when comparing two prospects that lead to losses. The experiments of Kahneman and Tversky (1979) have been seminal in the field of behavioural finance, and the results of their work strongly influence the way individuals are modelled in this paper.

Discount rates and risk premia are embedded in economic models where agents choose how to allocate their wealth into financial products according to their preferences, taking into account a set of assumptions about human behaviour such as more consumption is better than less consumption, individuals seek a balance between working hours and free time, and so on. Utility functions should therefore take into account all the factors that determine individual preferences to provide the best description possible. Hence the setup of the utility function can strongly affect the meaning and the outcomes of an economic model. Among the several utility functions available, power utility functions seem to have the most interesting set of properties in a compact functional form. The paper from Xie (2000) introduces an interesting specification of power utility that takes in consideration both risk aversion and the consumption level. In the Xie's framework the relationship between risk aversion and consumption is determined by a third parameter called rate of satiation. People with a low rate of satiation will in the long run end up with high consumption. This rate of satiation is very similar to perceiving a future return either as high or low according to the framework developed in the present paper.

Furthermore the research from Crossley and Low collects and brightly presents the whole set of the properties of power utility functions. Crossley and Low also demonstrates that panel data is inconsistent with power utility functions when intertemporal additivity is taken into account, and propose to abandon that utility specification to model the the utility of a representative investor. A further contribution is drawn from Sharpe (2006), that can be seen as the father of the aggregation approach. While aggregation is seen here simply as averaging, in latest book published by Sharpe the preferences of several individuals are brought together using a trading simulation. This description of the aggregated market can also be reconciled with the capital asset pricing model.

The third group of important contributors provides the quantitative tools that concretely allowed to apply the above mentioned set of theories to the market data. As first proposed by Äit Sahalia and Lo (1998), the risk neutral density can be estimated from option prices and then used to estimate nonparametrically the empirical pricing kernel of the market.

Furthermore the paper from Heston (1993), who first introduced stochastic volatility models in option pricing, is another cornerstone that allows to perform an estimate of the empirical pricing kernel. Heston (1993) allowed to describe the dynamics of option prices embedding also the dynamics of volatility. Using the Heston model it is indeed possible to deal with the volatility smile, a property of the dynamics of option prices and volatility that the standard Black and Scholes theory does not capture. The Heston model has been implemented as proposed in the insightful paper from Bergomi (2005). Together Heston (1993) and Bergomi (2005) constitute the framework in which the risk neutral density is estimated by Moro, Detlefsen, and Härdle (2007).

The present paper is structured in seven chapters. The next chapter deals with the estimation of the market utility Function, starting with the setup of the economic model in which the representative agent operates and then describing how to extrapolate the empirical pricing kernel from the market data.

Then chapter three will introduce the aggregation problem and provide all the explainations that are necessary to understand our assumption about investor's behaviour. Chapter four and five will then introduce the two dif-

ferent numerical algorithms that are used to solve the aggregation problem. These two mechanism differ from each other both in the way the final solution is computed and in the way the prior allocation is chosen. Although the first algorithm is more straightforward, the second algorithm delivers results that are less affected by parameter choices.
Chapter six collects the results of our computation and thoroughly describes the new insights that this work opens for the definition of particular investor profiles, and more generally for the modelling of individual preferences.
The final chapter closes the paper and wraps up in a critical perspective all the main conclusions about the aggregation approach and the algorithm presented in this paper.

Chapter 2

The Market Utility Function

The derivation of an estimate of the market utility function is performed by Moro, Detlefsen, and Härdle (2007) combining risk neutral asset pricing theories with a simple economic model. The basic economic model is drawn from the work of Cochrane (2001) and introduces a representative agent that chooses the amount of his or her endowement to be allocated in investment (DAX Index) in each period, in order to maximize its utility over time. Once this framework has been built, it is possible to express explicitly the relationship between the market utility and the market pricing kernel. In turn, the pricing kernel is estimated from the quotes of option contracts (ODAX Contracts), assuming that they can be modeled according to Heston (1993).
The present chapter first describes the features of the economic model, and then shows how to use risk neutral asset pricing in order to compute the pricing kernel and the utility function of the representative agent. The final and most consistent part of the chapter explains how the empirical pricing kernel that has been provided for this paper has been computed.

2.1 Theoretical Background

2.1.1 The Equilibrium Model

The economic model adopted for the purposes of this work is inspired by Cochrane (2001) and describes the preferences of the representative agent within two periods in a time discrete framework.
A representative investor, a single entity that embodies the aggregate market, earns in period 0 an amount e_0 and in period T an amount e_T. This endowement can be allocated either in consumption today or invested in an

asset traded at a price P. The asset will yield a payoff $\upsilon(X)$ in the following period and the investor then chooses the amount assets to purchase ξ according to the optimization problem in formula 2.1.

$$\max_{\xi} U(C_0) + \beta E^p [U(C_T)] \tag{2.1}$$

s.t.
$C_0 = e_0 - P_0\xi$
$C_T = e_T + \upsilon(X_T)\xi$

The solution to the problem is easily found by substituing the constraints into the maximization problem 2.1 as shown in formula 2.2, and then setting the first order derivative with respect to ξ to be equal to zero, as in equation 2.3.

$$\max_{\xi} U(e_0 - P_0\xi) + \beta E^p [U(e_T + \upsilon(X_T)\xi)] \tag{2.2}$$

$$-P_0 \cdot U'(C_0) + \beta E^p [\upsilon(X_T) U'(C_T)] = 0 \tag{2.3}$$

Rearranging the term of 2.3 it is possible to write P_0 in an explicit form the basic pricing equation 2.4 can be obtained. According to the basic pricing equation, the value of the security purchased at time 0 is equal to the product of the expected value of future payoffs and the marginal rate of substitution of consuption over time, discounted by the risk free factor β.

$$P_0 = E\left[\beta \frac{U'(C_T)}{U'(C_0)} \cdot \upsilon(X_T)\right] \tag{2.4}$$

Equation 2.4 shows also that the price of the security today depends on the optimal balance of the consumption levels over time for the expected payoff. The payoff $\upsilon(X)$ has a different interpretation in case the representative agent is investing in fixed income securities rather than in stocks, or even in more sophisticated products.
In this context it is assumed that the representative agent is actually investing in german equity, and therefore the payoff $\upsilon(X)$ of the portfolio will be equivalent to the payoff of the DAX index.

$$E\left[\beta \frac{U'(X_T)}{U'(X_0)}\right] = E\left[\beta \frac{U'(C_T)}{U'(C_0)} \| X_T\right] \tag{2.5}$$

To conduct an empirical analysis from equation 2.4, it would be necessary to dispose of data about consumption rather than financial data. An alternative is then to project the consumption on payoff levels, by taking the conditional expectation as shown in equation 2.5. The pricing equation projected on returns is then given in equation 2.6.

$$P_0 = E\left[\beta\frac{U'(X_T)}{U'(X_0)} \cdot \upsilon(X_T)\right] \tag{2.6}$$

Using equation 2.6 it will be possible to use the financial data about the DAX index, available in a more complete dataset.
Furthermore it shall be noted that the equation 2.6 has a very strong similarity with the basic risk neutral pricing equation in 2.10. The latter equation is introduced in the following section of the chapter and is a fundamental tool to price options and other derivative instruments that have complex payoff functions.

2.1.2 Risk-Neutral Asset Pricing

Under the definition risk neutral asset pricing a wide literature is comprehended, including all the theories based on the no arbitrage theorem, ranging from the capital asset pricing model to the Black Scholes formula. A wide set of theories, initially developed independently, leads indeed to the same conclusion that the price of any asset can be expressed by equation 2.10.
In order to understand fully the basic risk neutral pricing equation (2.10), it is worthwile to recall that the riskiness of the payoff can be embedded in the probability measure. This risk neutral probability measure can be interpreted as the price of an Arrow-Debreu security, a security that pays off one unit of numeraire if a particular state of the world is reached and zero otherwise. According to this interpretation q is also called state price density, and allows to price each unit of payoff for any possible future state. The expected value of a risky payoff using the risk neutral probability measure q will then be computed as in 2.7.

$$E^q(X_{t+1}) = \int q(z) \cdot \upsilon(z)\, dz \tag{2.7}$$

The key feature of the risk neutral probability measure is that, when taking the expectation in q, the expected value 2.7 discounted by the risk free discount factor β will exactly give the price as in equation 2.8.

$$P_0 = \beta \cdot E^q\left[\upsilon(X_T)\right] = E^q\left[\beta \cdot \upsilon(X_T)\right] \tag{2.8}$$

Let's now introduce the subjective probability measure p, a density that describes the actual probability of a given payoff to be realized, without embedding any correction for the risk. Equation 2.9 shows that using the Radon-Nykodin derivative $m(X) = \frac{q(X)}{p(X)}$ it is possible to convert the expected value taken on the risk neutral probability to get the expected value on the actual probability.

$$E^q(X_{t+1}) = \int \frac{q(z)}{p(z)} \cdot p(z) \cdot v\,(z)\,dz = m(X_{t+1}) \cdot E(X_{t+1}) \qquad (2.9)$$

Using this approach it is possible to derive the price of any asset with state dependent payoff, as shown in the risk neutral pricing equation 2.10. In this case the risk adjustment will be introduced by the term $E\left[\beta\frac{q(X_T)}{p(X_0)}\right]$.

$$P_0 = E\left[\beta\frac{q\,(X_T)}{p\,(X_0)} \cdot (X_T)\right] \qquad (2.10)$$

Not only do the equations explained above allow to understand better the meaning of the pricing kernel, but it is also possible to apply the risk neutral asset pricing into the economic model presented in the previous section.

2.1.3 Derivation of the Market Utility Function

Two approaches that allow to derive the basic pricing equation have been introduced. The relation between these two ideas is easily found combining the equation 2.6 from the Cochrane model and the equation 2.10 from the risk neutral asset pricing theory in order to obtain equation 2.11.

$$m(X) = \beta \cdot \frac{U'(C_T)}{U'(C_0)} \qquad (2.11)$$

After grouping the discount factor and the marginal utility in time 0 into a single constant factor k, it is then possible to define the utility function as the integral of the product of the pricing kernel with a constant term.

$$U(C_T) = k \cdot \int m(z)\ dz \qquad (2.12)$$

Equation 2.12 will then allow for the computation of the market utility function once the estimate of $m(x)$ has been performed.

2.2 Estimation of the Pricing Kernel

To estimate the pricing kernel nonparametrically it is first necessary to perform the estimate of the risk neutral distribution q and the subjective distribution p, and than take the ratio of the two.
As previously mentioned, the q density is the price of an Arrow-Debreu security. Although such securities are not traded, the same information about the pricing of the future states can be estimated or approximated from the

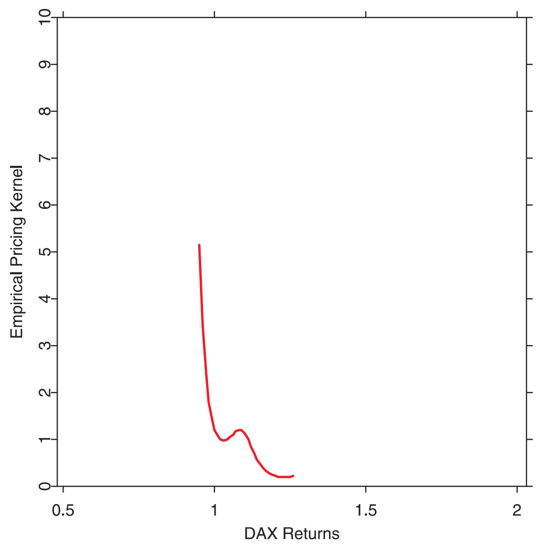


Figure 2.1: Plot of the pricing kernel, as ratio between p and q, estimated on date 30-06-2004

quotes of traded options contracts.
Making some strong assumption about the price dynamics of the underlying asset it is possible to obtain the risk neutral density in a closed form. For instance if options are priced under the Black-Scholes-Merton model, the risk neutral density will be simply a lognormal distribution. For more complex stochastic processes, the state price density can be computed only numerically.
Another approach is to specify the prior state price density in a parametric form and then proceed recursively to improve the prior by minimizing the pricing error.
Finally, a third alternative method is the nonparametrical estimate using the tools developed by Äit Sahalia and Lo (1998).
The following two sections implement a pricing kernel estimate based on the Heston model for the risk neutral density and GARCH for the subjective density.
Figure 2.1 contains a plot of the empirical pricing kernel estimated on date 30-07-2002. This curve is obtained as the ratio between the risk neutral and the subjective density, that are in turn displayed in figure 2.2.

2.2.1 Estimate of the Risk Neutral Density

The empirical pricing kernel used in this work has been computed as by Moro, Detlefsen, and Härdle (2007).
They choose to model option prices using the stochastic volatility model of Heston (1993), that includes the impact of stochastic volatility on the market quotes and allows a more realistic representation of the price dynamics. This choice is justified in the light of the research conducted by Bergomi (2005), that shows that in this context the dynamics are more important than a perfect fit.
Furthermore the Heston models is nothing more than the transposition into a continuous time environment of the GARCH model, that will be used to estimate the actual density p of the DAX Index in the following section.
The stochastic volatility model of Heston consists of two differential equations 2.13 and 2.14.

$$\frac{dS_t}{S_t} = r \cdot dt + \sqrt{V_t} \cdot dW_{1,t} \tag{2.13}$$

Equation 2.13 describes stock returns by normal innovation with stochastic variance, where r is the risk free interest rate.

$$dVt = 2 \cdot (\eta - V_t)dt + \theta\sqrt{V_t} \cdot dW_{2,t} \tag{2.14}$$

Equation 2.14 describes the stochastic variance process as a square root diffusion.
A set of parameters with economic interpretation are then defined.
The two Wiener processes W_1 and W_2 have correlation ρ. This parameter models the leverage effect between variance growth and stock growth. The choice of ρ affects the skew effect.
The volatilty of variance θ controls mainly the kurtosis of the distribution of the variance. Its choice affects the shape of the volatility smile.
The parameter η represents the long variance, as to say the level around which the short variance V_t oscillates. The speed at which the return variance returns to the long variance level has been set equal to 2, as Bergomi (2005) proposes and motivates.
Finally the parameter V_0 is the short variance at which the process starts.
Given the above described model for the underlying prices, it is possible to derive that equation 2.15 is the semi-closed definition of the call price from Carr and Madan (1999).

$$C(K,T) = \frac{exp\{-\alpha ln(K)\}}{\pi} \int_0^\infty exp\{-ivln(K)\}\psi_T(v)dv \tag{2.15}$$

$\psi(v)$ is a function of $\phi(z)$, the characteristic function of $log(S)$.
Fitting the model's implied volatility IV to the implied volatility of market option prices IV^* the parameters in $\phi(z)$ can be calibrated. The calibration can be performed with a simple least square method, that minimizes the term shown in equation 2.16.

$$SE = \sqrt{\sum_{i=1}^{n} \frac{1}{n}(IV_i - IV_i^*)^2} \tag{2.16}$$

The dataset that has been used by Moro, Detlefsen, and Härdle (2007) includes only call prices with six months time to maturity. Only contracts with strike prices within a range of $\pm 50\%$ of the at-the-money value have been used, in order to remove illiquid and therefore most likely mispriced observations from the sample.
Once the charcteristic function $\phi(z)$ of $log(S_T)$ is known, it is possible to recover the density of $L_T = log(S_T)$ using the Fourier inversion presented in the equation 2.17.

$$f(l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\phi_T(t))dt \tag{2.17}$$

The estimate of the risk neutral density of $S_T = exp(L_T)$ is finally shown in equation 2.18.

$$q(l) = \frac{1}{l} f(log(l)) \tag{2.18}$$

The estimated risk neutral density on date 30-07-2002 is plotted in figure 2.2 together with the estimated subjective density.

2.2.2 Estimate of the Subjective Density

For the estimation of the subjective density a wide array of methods are available. These methods can be efficiently classified according to the type of data used, either historical data or simulated data.
The former cathegory would comprehend all those techniques that can be applied to describe a sample of historical observation. It could for instance be possible to fit some pre-specified functional distribution to the data, or alternatively to apply histograms, kernel density estimators as well as any other tool one could prefer.
On the other hand the approaches that belong to the latter cathegory would then use the findings about the historical data to model the features of the capital markets, for instance with an ARMA, ARCH or GARCH model. The

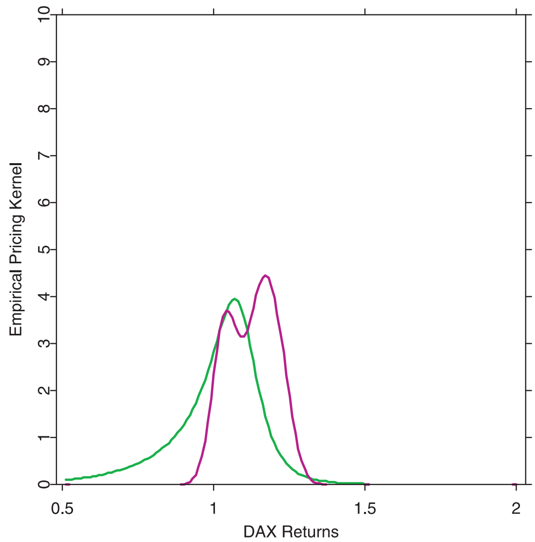


Figure 2.2: Plot of the risk neutral (pink) and subjective (green) densities estimates on date 30-06-2004

new model will then be used to generate a very large sample of simulated data, and the distribution of this outcome can be explored easily with any descriptive technique.
As mentioned in the previous subsection, the Heston model suggests that the underlying asset can be modelled with the GARCH model. The formulation of the GARCH model proposed by Franke, Härdle, and Hafner (2004) is given in the equations 2.19 and 2.20.

$$R_t = \sigma_t Z_t \tag{2.19}$$

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{2.20}$$

Given that R_t is the log return correponding to observed DAX quotes, once the conversion of the returns into prices is done it is possible to fit the parameters of the model to historical DAX quotes and use the GARCH model to run Monte Carlo simulation and generate the sample of simulated returns. The non parametric kernel density estimation of the simulated DAX absolute returns will then allow to plot the subjective density in figure 2.2. From the risk neutral and the subjective densities it is finally possible to estimate non parametrically the pricing kernel in figure 2.1.

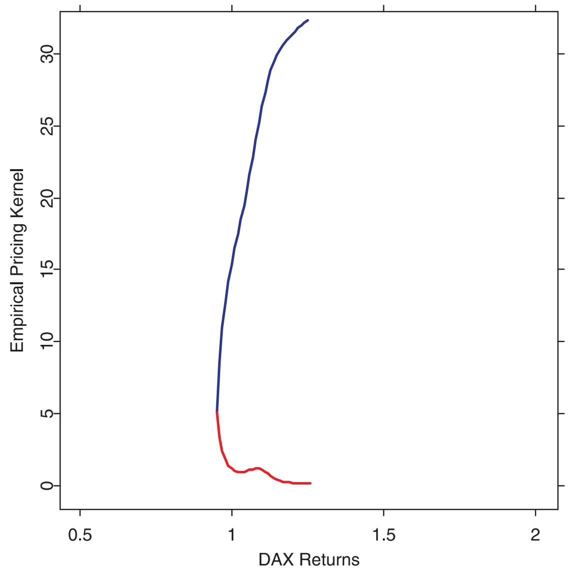


Figure 2.3: Plot of the estimated pricing kernel (red) and the estimated market utility function (blue) on date 30-06-2004.

2.3 The Estimated Market Utility Function

The previous sections of the chapter have shown how the pricing kernel and the utility function are tightly related. Furthermore a procedure has been presented that allows to compute a non parametrical estimate of the pricing kernel from the option prices in a specific date and the historical returns of the underlying asset. Once the estimate of the pricing kernel in figure 2.1 is derived, the relatioship between pricing kernel and utility holds as shown in equation 2.12. In the framework considered here, once the non parametric estimate of the pricing kernel is available, the non parametric estimate of the market utility function can be obtained by taking the cumulative sum of the pricing kernel. The estimate derived this way is plotted in figure 2.3 together with the estimate of the empirical pricing kernel.

Observing the plots it is possible to see that the empirical pricing kernel as well as the utility function are not monotone. This feature is pretty peculiar because it implies that no simple functional specification (i.e. logarithmic and power utility specification) can be used to model effectively all the properties of the preferences of the representative agent.

The pricing kernel represents the price per unit of return, and it is nothing else than the first derivative of the utility function. Therefore the pricing ker-

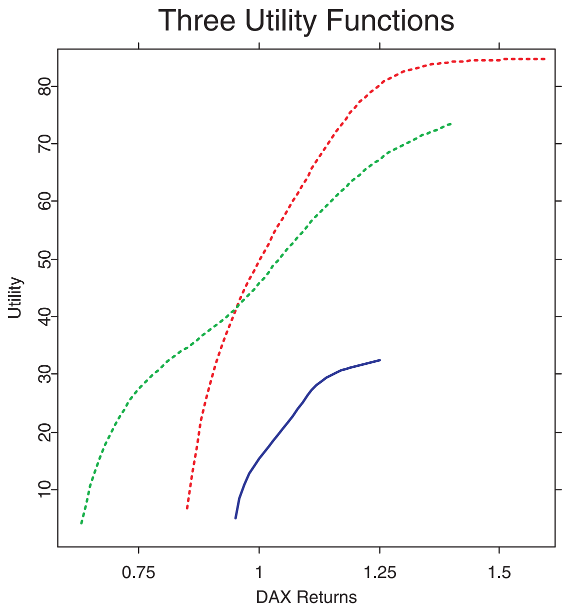


Figure 2.4: Plot of the estimated market utility function on date 24-03-2000 (red), 30-07-2002 (green) and 30-06-2004 (blue).

nel represents the marginal utility on returns. It still holds that the marginal utility is decreasing for higher returns, but it doesn't decrease in a continuous fashion. Indeed around some values, the marginal utility is again increasing and a sort of a hump can be observed. Therefore it is not only true that an extra unit of consumption is more valuable when the agent is poor rather than when the agent is rich, but it is also more valuable for certain states that are located between these extremes condition. In the case of figure 2.3 it can be noted that for returns above one (positive returns) the utility curve becomes again steeper, and the marginal utility increases and then decreases slowly again.
Although this feature is pretty puzzling, as already noted by Brown and Jackwerth (2005) it has been documented for several different geographical markets and different time periods.
In figure 2.3 three utility functions in three different periods are estimated and presented. The estimated market utility in year 2000 is concentrated on high positive returns. This implies that the market would have attributed with a drastically low utility negative returns. The utility function estimated before the implosion of the IT economy clearly portraits an optimistic outlook on financial markets. In this case no hump can really be observed, and marginal utility is always decreasing, although not constantly.

On the contrary in July 2002 the stock markets were experiencing a recessive trend, due to a relatively higher political instability and its implications on oil prices. The pessimistic feeling about financial market can be observed, and the perception of utility is relatively high even for negative returns.
Finally, it can be observed that in 2004 the market conditions have reached a more stable condition, and it is uncertain whether the german capital markets will grow or shrink, although no big shock is expected. The utility function in this period can therefore be estimated only for a more concentrated set of future states, since not many observations are available for extremely high or extremely low returns. All the future states are indeed associated with generally lower levels of utility than it has been observed in 2000 and 2002.

Chapter 3

The Aggregation Problem

The new frontier explored in the present work is a method to describe and classify the investors in the capital market. Similar approaches have been already pursued by Sharpe (2006), who has first proposed a derivation on the pricing kernel based on the aggregation of the preference sets of two agents is performed via trading simulation. To the extent covered in this work, the aggregation technique presented here and in the next chapter states that for a given utility level the correspondent market return is equivalent to the average of the levels of returns expressed by each class in relation to the given utility level. By averaging these implied returns, it is also assumed that the outcome of the market aggregation is similar to what a social planner would do in order to maximize the utility of all the participants in the economy. This assumption is not proven to be true, since the market doesn't behave as a social planner. Nevertheless it will be shown in chapter six that the procedure makes it possible to reconcile sensefully the micro and the macro models into an holistic theory.
The present chapter is divided into two sections. The first describes the microeconomic model proposed for the individual preferences, while the second one deals with the reconciliation issues between the micro and the macro level.

3.1 Utility Specification Based on Two Attitudes

Empirical studies demonstrate that agents change their perception of future returns under different circumstances. More precisely, in a scenario with low level of wealth, a unit of lost consumption will reduce the utility of the agent much more drastically than it would do in a wealthy scenario. Conversely, an

extra unit of consumption will create much more utility in the poorer scenario rather than in the richer scenario. To reflect this property the utility function should become flatter and flatter for higher return.
Furthermore, as Kahneman and Tversky (1979) observe, there is a number of irrational elements that justify the use of different local utility functions, according to the different future states considered.
Each investor is assumed to have two utility levels, representative of his or her attitudes towards different future states, perceived in different ways. The assumption brought forward is that the investor will undertake the attitude that relates each future state to the highest level of utility achievable, among two levels of utility associated with two attitudes.

3.1.1 High and Low Attitudes

The empirical evidence collected by Kahneman and Tversky (1979) shows that choices among risky prospects exhibit several features that are inconsistent with some common assumptions made in the utility theory. Their results constitutes the starting point of behavioural economics, a branch of the economic research that flourished in a large amount of influential publications over the last two decades. Among the most striking findings, it is proven that people underweight those outcomes that are merely probable in comparison with outcomes that are obtained with certainty. This tendency is called the certainty effect, contributes to risk aversion in choices involving sure gains and to a risk seeking attitude in choices involving sure losses.
Furthermore, it is found that the reflection of prospects around 0 reverses the preference order. Although it is not suprising to observe that when all the payoffs considered are positive individuals are risk averse, it is interesing to observe that risk aversion does not always hold. In fact it is observed that when all payoffs are negative, most individual considered a risky prospect superior to a certain loss, although the expected value of both prospects was the same.
Thus, the evidences from Kahneman and Tversky (1979) are incompatible with the notion that certainty is generally desirable. Rather, it appears that certainty increases the aversiveness of losses as well as the desirability of gains. The reversal of preferences due to the dependency among events is particularly significant because it violates the basic supposition of a decision-theoretical analysis, that choices between prospects are determined solely by the probabilities of final states.
A last important finding is that people generally discard components that are shared by all prospects under consideration. This tendency, called the isolation effect, leads to inconsistent preferences when the same choice is pre-

sented in different forms.
To the extent covered by the present work, the empirical data collected by Kahneman and Tversky (1979) motivates a specification of the individual utility functions based on different preference sets for different conditions.
As shown in figure 3.2 the utility function presents a change in concavity for future states around one. This implies that the market utilty function is not homogeneous and it doesn't resemble the shape of any simple functional specification. Nevertheless, it can be well approximated using two curves rather than one. The two curves plotted into figure 3.2 are two power utility functions, and, as it will more clearly stated further in this section, they are representative of two different attitudes in appreciating future states. In this peculiar case, it is interesting to notice that the hump in the market utility function is located around one, where investors could be particularly sensitive to the difference between losses (returns below one) and gains (returns above one). This consideration alone opens interesting possibilities, although it does not prove that the reflection effect alone can explain the shape of the market utility function. Furthermore, as shown in figure 2.3, the change in concavity appears around one only for the market utility observed in 2002, when the market underwent a bearish phase. A closer look to the the utility functions for years 2000 and 2004 in figure 2.3 is indeed sufficient to state that the hump in the market utility does not necessarily discriminate between the perception of absolute losses and absolute gains. Nevertheless looking at the hump in the market utility function in 2000 and 2004 it is possible to mark a difference between lower and higher returns. The so called reflection effect might not hold among absolute positive and absolute negative market returns, but it holds that those returns that are higher or lower returns than benchmark are perceived in two different ways.
The observed data makes it interesting to explore the assumption of different utility levels that depend on the future states, and composite utility functions have already been suggested by Kahneman and Tversky (1979) as a possible way to deal with the reflection effect described previously. In this work it is assumed that each individual investor can undertake only two attitudes. One attitude reflects the utility of those states that are associated with lower returns, while the other describes the preference set of higher returns. Several classes of investors can then be defined and a specific high attitude is what differentiates each class from the other. On the other hand the low attitude is assumed to be described by the same function for all classes of investors.
For every future state, there are now two possible utility levels $U_L\left(X_T\right)$ and $U_H\left(X_T, shift_i\right)$, where the $shift_i$ is a shifting parameter in the utility specification that corresponds to the investor type i.
As noted by Sharpe (2006) the allocation choices are determined by several

factors. Different investment choices should be done by investors with different positions in terms of initial holdings of securities, levels of wealth, as well as geographic location, home ownership, profession, etc. Furthermore investors have different preferences, in terms of feelings about risk, present versus future gratification, and so on. Finally investors might make different predictions, in terms of feelings about the probabilities of alternative future outcomes. All these elements justify the definition of several classes. Since it is would be very hard to distinguish between all those factors and model each of them separately, the use of shifts allows to model different categories with a reasonable approximation.
Sharpe (2006) also offers the idea of using the power utility function as it will be also presented here, as well as the idea of introducing individual investor with kinked utility function. As a short about the power utility function, it is worth mentioning that it has been criticized by Xie (2000) as not suitable to model all the properties of the aggregate market. Nevertheless no evidence is indicating that such formulation is not able to describe correctly individual preferences. Consequently, the utility functions for both attitudes can be modelled as shown in equations 3.1 and 3.2, where $i = 1, ..., N$ defines N investor types.

$$U_L = a_L \cdot (X_T - c_L)^{d_L} + b_L \tag{3.1}$$

$$U_H = a_H \cdot (X_T - c_H - shift_i)^{d_H} + b_H \tag{3.2}$$

The total utility of an investor of type i over all the possible state is defined by equation 3.3.

$$U_i(X_T) = max(U_L(X_T), U_H(X_T, shift_i)) \tag{3.3}$$

If it is true that for very low returns, most investors follow the low attitude while for very high returns, most investors follow the high attitude, then it must hold that the left tail of the estimated market utility function will be given by the aggregation of investor in the low attitude. In the same way, the right tail of the estimated market utility function will be given by the aggregation of investors in the high attitude. Therefore the parameters of the U_L and U_H functions should be fitted on the tails of the estimated market utility function.
The problem is now to define those tails better. The criterium proposed here is simply to look at the density of the DAX returns. In figure 3.1 the tails are defined in order to include the lowest twenty percentiles (0-20) of the DAX returns density for the left tail, and the highest twenty percentiles for the right tail (80-100).

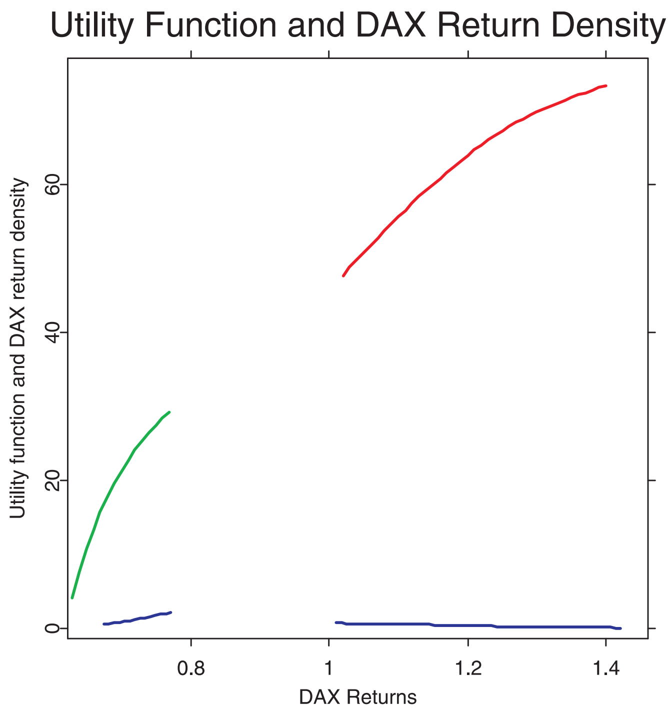


Figure 3.1: Tails of the estimated market utility function (green and red) based on the lower and upper twenty quantiles of the DAX return density (blue). Date 30-07-2002

Defining the tails on the basis of the twentieth percentile of the DAX return it is possible to estimate the parameters of the two utility levels plotted in figure 3.2, where the parameter $shift_i$ is assumed to be equal to zero.
It is worthwile to show that the estimated coefficients do not change substantially if different tails are chosen, as it is possible to observe in table 3.1. The above part of table 3.1 contains the estimated coefficients for the low attitude utility, assuming tails of ten, twenty and fifty percentiles of the DAX return density respectively in the first, second and third line. In the bottom part of the table, the same coefficients in the same order are also given for the high attitude utility with null shift parameter. Although these coefficients do not present radically different features according to the chosen tail, it will be shown in chapter five that the choice of the tail has some impact on the final result of the aggregation.

As anticipated in the beginning of this chapter, the aggregation procedure is presented here as a weighted average. Before going further in the explaina-

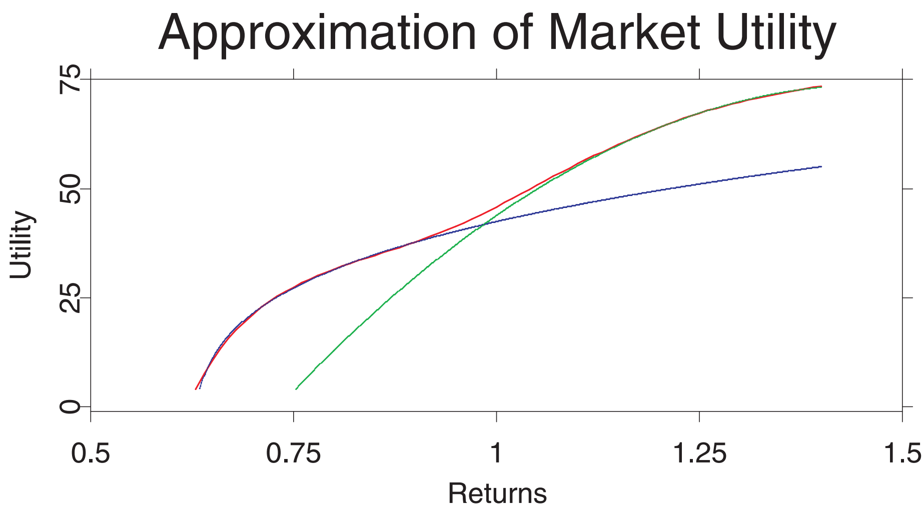


Figure 3.2: Estimated market utility function (red), fitted low individual utility level (Blue), fitted high individual utility level (green). Date 30-07-2002

Tail	a_L	b_L	c_L	d_L
10-10	119.6	-50.84	0.601	0.219
20-20	80.58	-20.57	0.626	0.25
50-50	220.5	-160.25	0.59	0.09

Tail	a_H	b_H	c_H	d_H
10-10	-105	74.42	1.532	2
20-20	-134	73.92	1.476	2
50-50	-100.5	74.7	1.515	2

Table 3.1: Estimated coefficients for utility levels fitted to ten-, twenty- and fifty-percentile tails of the market utility function.

tion, it is worth recalling that utility is not expressed in any specific unit of measure. The value given by an utility function can be used to compare and rank preferences, but it does not have any univocal meaning as a measure. Furthermore utility itself is not an additive concept, and the joint utility of two agents is not necessarily equivalent to the sum of the utilities of all the agents. These considerations are sufficient to rule out the possibility of achieving an aggregation of individual utilities simply by adding or averaging utilities.

Nevertheless, if the utility function establishes a biunivocal relationship between returns and utility, it is also possible to invert this relationship and go backward from utility to returns. In order to obtain an aggregated market utility function, the total individual return function 3.4 should be defined as the inverse function of the total utility 3.3. Those implied returns are additive and the aggregation can be performed by averaging, in the same way as the returns of a portfolio can be obtained as the the weighted average of the returns of all its components.

$$X_i(U) = U_i^{-1}(X_T) = min(X_L(U), X_H(U, shift_i)) \tag{3.4}$$

The implied return functions X_L and X_H are defined by the inverse of the individual utility function respectively for both the high and the low attitude. The functional specification of the implied return functions for the high attitude and low attitude utilty level can be found in equations 3.5 and 3.6. In equation 3.6 the parameter $shift_i$ is now shifting the implied return curve of the high attitude up and down, so that the different modes in which different classes of investors perceive future returns are captured.

$$X_L(U) = (\frac{U - b_L}{a_L})^{\frac{1}{d_L}} + c_L \tag{3.5}$$

$$X_H(U) = (\frac{U - b_H}{a_H})^{\frac{1}{d_H}} + c_H + shift_i \tag{3.6}$$

Figure 3.3 gives a graphical representation of the market return function, plotted in red, and the two implied return functions for the low attitude and high attitude. It has already been shown in figure 3.2 that the peculiar humped shape of the market utility can be replicated by the two utility functions for two attitudes, and also the shape of the market return can be well approximated using two implied return functions. In figure 3.3 it is also shown in pink how the implied return function for the high attitude can be shifted up and down according to different choices of the $shift_i$ parameter.

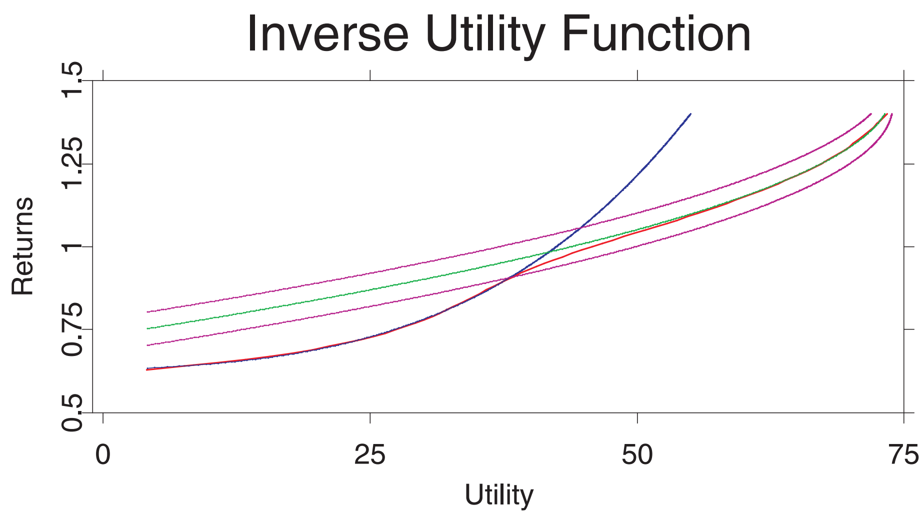


Figure 3.3: Estimated market return function (red), low attitude return function (blue), high attitude return function (green), two shifted high attitude return functions (pink). Date 30-07-2002

3.1.2 Switching Points

In the previous sections it has been explained how each investor class differentiates itself from the others using a shift parameter, that affects both the utility level and the implied return of the investor in the high attitude. Although shifts perform well their duty, allowing at the same time a simple formulation, they lack of a clear economic interpretation. At this point it is not easy to realize what is implied by a given shift parameter, nor is any criterium for the choice of these switching point given. Furthermore particular care should be put in dealing with the mere fact that, for some shifts, the implied return functions $X_L(U)$ and $X_H(U)$ don't even have an intersection and as a result there will be investors who are only following the low or the high attitude. If the choice of the shift is not done carefully, the risk is to define several classes of investors that follow the low attitude over the whole range of future states considered. In this case it would be problematic to deal with these different entities that are actually identical and should be grouped in a single class.
To avoid both these theoretical and computational issues, it is then convenient to introduce a new concept, the switching point. A switching point is simply the intersection between the implied return function $X_L(U)$ and $X_H(U)$. On one hand switching points have a straightforward economic in-

terpretation, on the other hand they can be used to define the set of shifts. As stated in the definition 3.7, in a switching point the investor is indifferent between the two attitudes, since both provide the same utility. In the switching point the individual follows a more rational approach, and he or she is not affected by the reflection effect discovered by Kahneman and Tversky (1979) and mentioned above, being the utility in the switching point not affected by the attitude chosen. This interpretation of the switching point also attributes a different kind of reflection effect to each investor class. If the utility functions are defined according to the power utility specification and the switching point is given, it follows that for the future returns that lie below the switching point, the low attitude is dominant. Conversely, for the future state above the switching point, the high attitude will have an higher utility than the low attitude. It also holds that the larger the distance from the switching point, the wider the gap between the two utility levels.

$$X_i^{SW} \,||\, U_L(X_i^{SW}) = U_H(X_i^{SW}, shift_i) \tag{3.7}$$

where

$$U_L(X_i^{SW}) > U_H(X_i^{SW}, shift_i) \text{ for } X_T < X_i^{SW}$$

and

$$U_L(X_i^{SW}) < U_H(X_i^{SW}, shift_i) \text{ for } X_T > X_i^{SW}$$

Beside the easier economic interpretation, the other reason that motivates the use of the switching points is the possibility to derive a shift corrisponding to a switching point. Switching points and shifts should be chosen to improve an economic sense to the aggregation, for instance by restricting the interval of future states where changes between attitudes can occur, and ruling out the possibility that some investor classes follow only an attitude. Once the switching point has been defined, the shift can be derived in few steps. From the definitions of the implied return function in equations 3.5 and 3.6 and from the definition of switching point 3.7, it is known that under the high attitude the implied return in a switching point can be equivalently derived using either the high utility level or the low utility level, as in equation 3.8.

$$X_H(U_H(X_i^{SW}, shift_i), shift_i) = X_H(U_L(X_i^{SW}), shift_i) \tag{3.8}$$

In parallel it also holds that in the switching point the implied return can be identically derived either from the low or the high implied returns, as in equation 3.9.

$$X_i^{SW} = X_L(U_L(X_i^{SW})) = X_H(U_L(X_i^{SW}), shift_i) \tag{3.9}$$

Given the specification of the inverse return functions based on the power utility function as in equation 3.6, it is possible to use the additive property of this function to write the shift explicitly, as shown in the equation 3.10.

$$X_H(U_L(X_i^{SW}), shift_i) = X_H(U_L(X_i^{SW}), 0) + shift_i \tag{3.10}$$

The terms in 3.10 can be rearranged in order to bring the shift to the left-hand side of the equation and leave all the other factors of the equation on the right hand side. Furthermore, according to equation 3.9, it is possible to substitute X_H with X_L and derive a more compact defininition of the shift, as shown in equation 3.11.

$$shift_i = X_L(U_L(X_i^{SW})) - X_H(U_L(X_i^{SW}), 0) \tag{3.11}$$

The derivation of the shift from the switching point allows to generate the whole range of implied return functions and utility functions for the high attitude level. In other words a grid of all the possible attitudes that investors might undertake is now available. The switching points and therefore the grid of investors classes can be adjusted to reflect the market moment and the different outlooks of future returns. Once this framework has been set up, it is then possible to focus on the central topic of this paper, finding the optimal distribution of investor among all these classes.

3.2 Setup of the Aggregation Problem

In the previous sections of this chapter all the theoretical tools to perform the aggregation have been already introduced, and this section finally presents the aggregation procedure. Given our set of assumptions about the behaviour of the individual investors, the goal is now to reconcile the micro model with the market data.

In chapter two the market utility function has been estimated using a non parametrical approach, and in the previous section of this chapter the implied returns have been defined as the inverse of the utility function. The final outcome of all these computations is the vector $\hat{X}_M = [u_M, x_M]$, a discrete definition of the market implied return function.

The starting point for the aggregation is the definition of the aggregate market return as the weighted average of all the individual implied return functions, as from equation 3.12.

$$X_M(U) = \sum_{i=1}^{N} \theta_i X_i(U) \tag{3.12}$$

Each weight θ_i represents the amount of investors of type i for i = 1, ..., N, as to say those investors who switch to the high attitude after the switching point X_i^{SW}.

Performing the aggregation as the weighted average of returns is absolutely plausible in the case of realized returns. Nevertheless, it shall be noted that the implied returns are not the returns that are earned by each investor class in the market. These returns are indeed drawn from the utility maximization of each investor class and represent an ideal return rather than a realized return. Therefore the whole aggregation problem presented here lies on the fundamental assumption that the market is a sort of black box that fullfills all the needs of their participants. There is no evidence against or in favour of this view of the market, but this approximation of reality must be clearly understood before going forward in the analysis.

The core of the aggregation problem is given in equation 3.13. The best fit Θ^* vector is the solution of a minimum square error minimization problem that reduces the difference between aggregated market returns $X_M(U)$ and $x_{M,j}$, the implied market returns estimated in chapter two.

Also here the θ_i for i = 1, ..., N are the weights for the investor classes and j is defined for j = 1, ..., m, where m is the number of rows of the vector $\hat{X}_M$.

$$min_{\theta_{1,...,N}} \sum_{j=1}^{m} [x_{M,j} - \sum_{i=1}^{N} \theta_i X_i(U_i)]^2 \tag{3.13}$$

st.

$$\theta_i \geq 0 \quad for \quad i = 1, ..., N$$

Due to the complex formulation of the total implied return X_i given by equation 3.4, and due to the high number of variables to be drawn from the constrained optimization problem stated in equation 3.13, the solution can not be obtained analytically. As a consequence, numberical approaches have to be introduced.

The numerical approaches presented in the next chapter are drawn from random search algorithms and machine-learning algorithms, more specifically from the boosting technique. Although these kinds of algorithms normally find an application in frameworks that are out of the focus of this work, from a statistical perspective they can be seen as minimization of a convex loss function over a convex set of functions. As shown in the following two chapters these numerical algorithms are suitable to solve the aggregation problem.

Chapter 4

A Simple Search Algorithm

As thoroughly described in the previous chapters, two ways to compute the market utility function are now available. The first approach is the estimation of the market pricing kernel using the GARCH and the Heston models explained in chapter two. The other approach attempts to aggregate together the preferences of several investors. As shown in the last section of chapter three, the aggregation cannot be easily solved and the present chapter introduces a numerical approach that can be used to solve the aggregation problem described in chapter three. The objective of this problem is to minimize the difference between these two representations of the market. The parameters that have to be chosen in order to achieve the best fit are the weights of each investor class in the market.
The configuration of this constrained optimization problem suggests to use a Bayesian approach. The starting point of the simple search is that the investor types in the market are uniformally distributed. This prior distribution is then improved by swapping the position of a single investor from a randomly chosen type to another randomly chosen type and when the new distribution improves the fit the change is kept.
Two main findings about this approach should be mentioned. First it is noted that the solution is influenced by the number of investor classes that are to be considered, and by the granularity of the discretization of the weights. Furthermore a third parameter is needed to avoid the overfitting error and obtain a smooth solution.
In the first section of the chapter a thorough description of the numerical algorithm explains how the procedure computes the solution, while its most striking features are then presented in the second section.

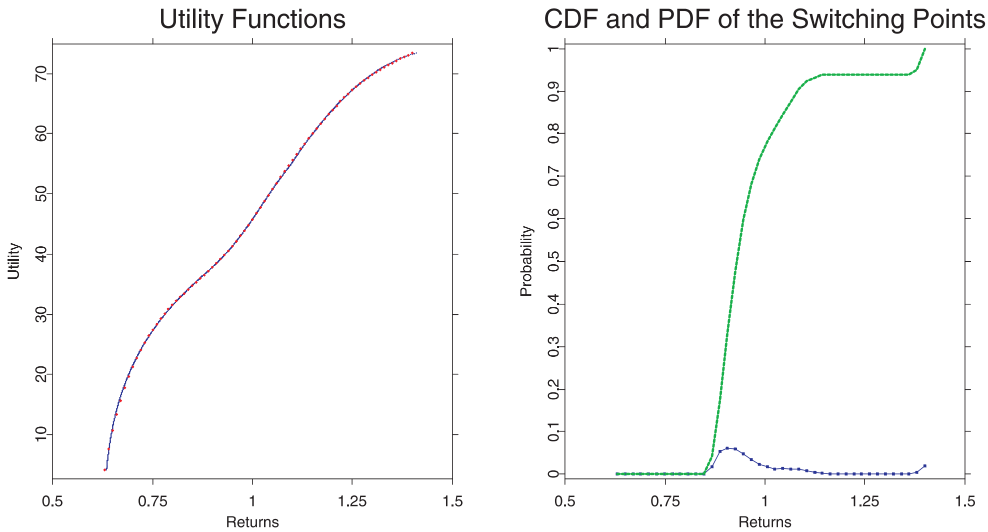


Figure 4.1: Left: Aggregated (Blue) vs estimated (Red) market utility function. Right: estimated allocation (Blue) and cumulative sum (Green) of the weights. Date 30-07-2002

4.1 Description

The simple search approach starts from the assumption that the whole market can be discretized in a given number of weights, each of them representing a fraction of the whole public of investors. As a starting point the weights are allocated among the possible classes using a uniform prior distribution. The switching points that determine the classes of investors are chosen within a range of returns that corresponds to the range of returns for which the market utility estimate is available. It is worth recalling that in chapter two the estimate of the market utility function has been performed only for a given range of market returns. More precisely, the estimated market utility function in 2002 is defined only for returns between 0.64 and 1.4 (returns from 36% to + 40%), in 2000 the range is from 0.85 to 1.35 (returns from -15% to +34%) and in 2004 the range is from 0.95 to 1.19 (returns from -5% to +19%).
Three parameters have to be chosen, and these are the number of switching points (SP), the number of weights on each switching point (NN), and the number of iterations.
The initial allocation is defined by the number of switching points and the number of weights that have to be put on each switching point. Choosing a

number of switching points implies to choose a number of different investor classes to be included in the analysis. From a more formal point of view, the number of switching points is equivalent to the number of variables that have to be found, in order to minimize the mean square error between the aggregated and the estimated inverse market utility.

The number of weights for each switching point affects the granularity of the discretization of the whole market. More simply put, if the initial allocation consists of two thousands weights for each investor class, the discretization will then have higher granularity than it would be the case of twenty weights for each investor class. From a more formal point of view, the number of weights that are initially allocated to every investor class is the initial value of the variables for which the optimization problem has to be solved.

This initial allocation Θ_0 is then altered through the swapping procedure. As its name should suggest, this swapping procedure basically attempts to move from a switching point to another. In the practical implementation, the random numbers are generated using the pseudo-random number generator *rand()* of C++. Once two random numbers have been generated, the swapping procedure subtracts one weight from the investor class designated by the first random number, and then adds one weight to the investor class designated by the second random number.

After the swapping procedure generates this perturbation in the allocation of weights, the weighted average of the individual implied returns is recomputed using the new set of weights. The mean square error between implied market returns estimated with the pricing kernel approach and with the aggregation approach is then computed again, according to the definition 3.13. Finally, the fit using the new set of weights is compared with the fit previously achieved. The change is kept if the fit of the aggregated implied market returns is superior after the perturbation, and is memorized as Θ^*.

The above described process has to be iterated several times, until a Θ^* that delivers a very small mean square error is found. An issue arises when the mechanism that stops the algorithm must be chosen. As explained further in this chapter, the simple search approach does not allow to use only the mean square error minimization criterion. Simply telling the algorithm to keep swapping until it is not possible to improve the fit will lead to an overfitting problem. Therefore it is necessary to define a third parameter to achieve a proper smoothness. Concerning the final allocation Θ^*, it is indifferent whether to set a limit to the number of swapping attempts, or to set a limit to the number of swaps that are actually performed. Limiting the swap performed would allow a direct control of the number of perturbations performed, but limiting the overall number of attempts allows to avoid the trap of infinite iterations. Since using both limitations will make the comparison

of the estimated results more complicated, the use of a limit to the number of attempted swap is the best choice.
A smoothed distribution of the investors among the possible classes is shown in figure 4.1. The box on the lefthand side contains the estimate of the market utility obtained from aggregation (blue line) and the market utility obtained from the risk neutral asset pricing (red dots). The box on the righthand side shows how the weights are distributed over the range of returns considered (blue dotted line) and presents their cumulative sum (green line).
To obtain this estimate the number of switching points has been set equal to forty, while the prior allocation is uniform and puts two hundreds weights on each switching point, for a total of eight thousands weights.
The algorithms stops after one hundred thousands attempts to swap.
The interpretation of figure 4.1 may not be straightforward, and a deeper analysis of the results is given in chapter six. It should now suffice to mention that the most informative element in the plot is the blue line that represents the optimal weight allocation Θ^*. For each return, the blue line indicates the percentage of investors in the market that switch to a different utility level after that point. The present chapter focuses on the estimation of a proper allocation and the features that are relevant are now the smoothness of this curve and the quality of the fit.
It can be observed that the fit achieved using this approach is very good and the means square error between the aggregated and the market utility converges to 0.000451311.
The blue curve in the right box shows that the optimal weight allocation estimated under the above mentioned parameter choices delivers a smooth result and a single peak can be observed for returns equal to 0.9. Furthermore, a strange peak is generated on the right tail of this curve.
This tail can have an economic interpretation. Since a switching point the utility under both attitudes is equivalent, switching in the high extreme of the range of returns considered is equivalent to not switching altogether. It could be true that in the market there is a large amount of investors who never switch, but it could also be the case that this curve has been generated by the algorithm only to improve the the fit of the estimated optimal weight allocation Θ^*.
The former explanation is supported by the fact that using the market data of 2000 and 2004 this tail does not appear, as well as by the fact that this tail can also be observed when using another numerical approach introduced in chapter 5. For these reasons it could be possible that given the market condition of 2002, some investors will perceive all the returns in the range under the low attitue.
The originating elements of this tail should be found in the assumptions that

have been made about the range of switching points and in the specification of the individual utility. A wider range of returns and switching point would allow investors to switch also for higher returns, and therefore would allow to smooth the tail. Furthermore a different specification of the initial grid of utility levels as derived in chapter three would also lead to an allocation of weights with no such tail.

4.2 Sensitivity Analysis

As explained in the previous section, the algorithm requires the choice of several parameters: the number of switching points, the initial number of investors assuming a uniform prior distribution and the number of iterations that have to be performed.
The problem that is being solved does not have a unique solution and the parameter choice affects the final outcome of the computation. It is therefore critical to choose the right parameters in order to achieve a meaningful estimate of Θ^*, since both an uncritical and an excessively personalized use of this algorithm might flaw the meaning of the final outcome.
The parameter that has the strongest impact is clearly identifiable in the stopping criterion of the algorithm. Indeed, what can be observed is that the model tends to overfit when the number of swaps attempt is not limited to a certain amount. In other words the number of iterations improves the fit, but it also plays an important role as smoothing parameter of this simple search algorithm, and therefore it must not be set too high. Figures 4.3 and 4.2 present different estimates given different parameter choice, and in both of them for the lefthand column ten thousands runs are performed, a hundred thousands in the central column and a million in the righthand column. The plots obtained with ten thousands swap attempts present spikes that are caused by a lack of fit. On the contrary the spikes plotted in the third column are an effect of overfitting generated by the one million attempts to swap. When too many iterations are performed, the algorithm generates a very detailed solution, with an high number of peaks. Nevertheless, although these peaks contributes slightly to achieve a lower mean square error, these perturbations brought to the allocation of weights are not carrying any information that is found in market data.
It is sufficient to recall all the passages through which the original source of information has undergone to realize that the market utility estimate, as well as to recall how the grid of investors that has been generated according to the theory of Moro, Detlefsen, and Härdle (2007) to realize that the optimal

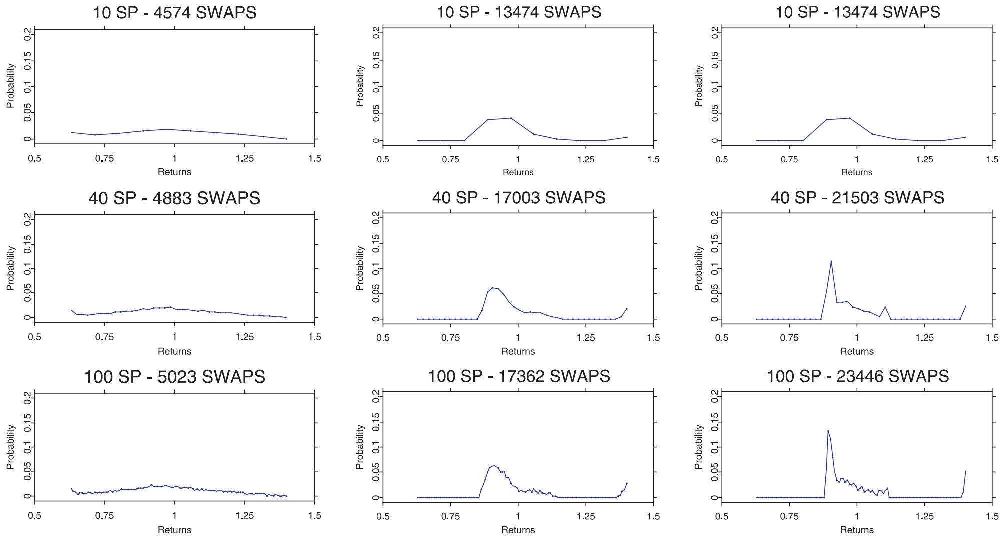


Figure 4.2: Sensitivity of the allocation of weights to the choice of the number of investor classes and number of iterations, date 30-07-2002

Iterations	10000	100000	1000000
SP = 10	0.0040603	0.000475816	0.000475816
SP = 40	0.00372617	0.000451311	0.000445253
SP = 100	0.00342665	0.000454054	0.000445802

Table 4.1: Mean square error of estimates using different number of iterations and number of investor classes parameters. Date 30-07-2002

allocation Θ^* computed with the algorithm is an approximation. There is no element to support of such a complex and extremely accurate description of the preferences of these investor classes, since this information has been lost in the many approximations already performed.
The next two subsections will provide a more detailed description of how the choice of the switching points and the choice of the number of weights affect the estimated Θ^*, and how these parameters can accentuate or soften the overfitting problem for increasing number of iterations.

4.2.1 Choice of the Switching Points

Among the three parameters that affect the accuracy and the smoothnes of the solution, the number of switching points is the most critical one from a theoretical point of view. At the same time, this parameter is the one that affects the final outcome less.
It has been already mentioned that the number of switching points corresponds to the number of investor classes that are considered. Furthermore the number of switching points also defines the complexity of the solution of the aggregation problem.
Although this parameter has so many implication, it is surprising to observe that the solution doesn't really change radically if more or less switching points are chosen. Figure 4.2 shows several plots obtained under different parameter choices. Each column displays the plots for a given number of iterations, namely ten thousands, a hundred thousands or a million iterations respectively in the left, central and right column. Each row contains the plots for a given number of switching points. The top row contains the plots assuming that the investor classes are ten, the central row when the investor classes are forty, and finally the bottom row contains the plots obtained when the switching points are one hundred. Above each plot the number of switching points and the number of swaps that are actually performed by the algorithm are given.
Each column shows three rather similar plots and this demonstrates that the number of switching points doesn't really affect the outcome of the solution. What changes remarkably is not really the allocation of the weights but the kind of plot obtained. It is enough to look once at the plots to realize that ten classes of investors might lead to an excessive approximation, while one hundred classes might be too many.
On the other hand it is also possible to see that the most important variable is the choice of the stopping point of the algorithm. As already mentioned before and as visible in the first column of figure 4.2, a small number of iterations leads to lack of accuracy in the estimate. As a result the plot will be too smooth.
At the same time too many iterations will generate the peaks in the plots of the third column of figure 4.2. These peaks are not a feature that is existing in the market data, since they are just a distortion introduced by the algorithm itself in order to marginally improve the fit.
It is interesting to notice that the allocation obtained under the hypothesis that there are only ten investor classes, a very small number of iteration is sufficient to reach an allocation where no furhter improvement is achievable. The plots are indeed identical when running a hundred thousand iterations

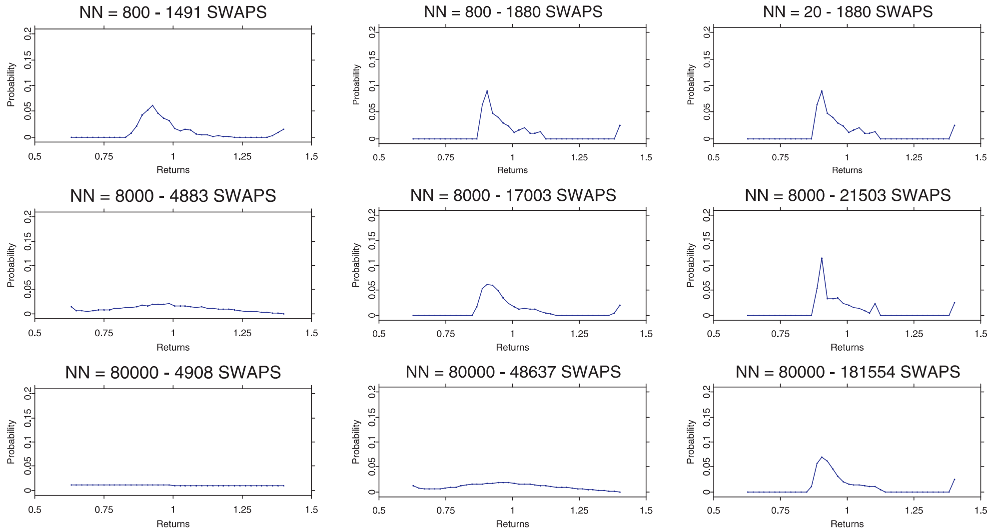


Figure 4.3: Sensitivity of the switching points density estimate to the choice of the number of initial investors and number of runs. Date 30-07-2002

Iterations	10000	100000	1000000
NN = 20	0.00056574	0.000476627	0.000476627
NN = 200	0.00372617	0.000451311	0.000445253
NN = 2000	0.00977071	0.00050017	0.00044639

Table 4.2: Mean square error of estimates for different number of iterations and number of investors. Date 30-07-2002

or when running one million iterations.
The table 4.2 also shows that the number of switching points does not necessarily increase the fit of the model, whereas it always holds that the bigger the number of iterations, the higher the fit obtained.

4.2.2 Choice of the Number of Weights

The last parameter left to examine is the number of weights that are initially allocated to each group of investor.
It has been already observed that the number of switching points is a parameter that defines the complexity of the solution of the optimization problem on

which the aggregation procedure is based. Although this parameter affects the smoothness of the estimated optimal allocation Θ^*, it does not really play a major role when looking at the accuracy of the results.
On the contrary it can be observed that the initial allocation of the weights among the switching points can affect significantly the smoothness and the accuracy of the estimated optimal allocation of weights.
After trying several types of initial allocations of weights it emerged clearly that a uniform prior would have been the best solution. In fact it has been observed that the iteration of the swapping procedure is not able to reshape any initial allocation. It has been seen that when starting from a normal or a triangular distribution of the weights, the simple search algorithm is not able to deconstruct the peak since it fails to find an improvement of the fit after a low number of swaps. The solution found this way is therefore too close to the prior. It is indeed necessary to start from an initial allocation Θ^0 that is sufficiently wrong to let the random swap improve the fit. Such sufficiently wrong prior can well be a uniform allocation.
Letting the number of switching points be equal to forty, figure 4.3 shows in a matrix of plots how sensitive the estimated allocation of weights is sensitive to different stopping points and to different values for the initial number of weights allocated to each switching points.
Each row of figure 4.3 displays the plots for a given number of weights per class. The row on the top displays the estimates using twenty weights per investor group, the central row contains the plots obtained using two hundreds weights per investor class, the bottom row contains the plots that have been obtained using two thousands weights per investor class. On each column of the matrix are displayed the outcomes obtained letting the simple search algorithm run up to a given number of iterations. On the left column the swap attempts performed are ten thousands, in the central column the attempted swaps are a hundred thousands, while for the right column the algorithm has performed one million iterations.
When comparing the optimal allocations obtained for different amounts of iterations, it is possible to see how the outcome changes from notched for ten thousands iterations to smooth for a hundred thousands iterations and eventually exhibits big peaks for a million iterations. This succession notched-smooth-peaks is particularly evident in the second row, for the two hundreds weights per investor class case, and shows once again how the choice of the number of iterations can both improve the fit and lead to overfit.
A comparison of the plots of figure 4.3 along each column would then show that for increasing amounts of weights, as to say for a higher granularity of the total amount of investors, the number of swaps required to achieve a smooth solution increases. Not only for an higher number of weights more

iterations have to be performed in order to achieve a smooth solution, but also the overfitting problems is delayed and would appear again for an higher number of iterations.

Looking at the plots along the main diagonal of the matrix it is possible to identify a common shape of the final outcome, representative of three well balanced sets of parameters.

The mean square error computed for several combinations of number of iterations and number of weights per investor class in table 4.2 offers another insight on the interplay between these two parameters. Each row of table 4.2 presents the final mean squar error obtained running the simple search algorithm using different values for the initial number of weights per investors, namely twenty weights per class in the top row, two hundreds weights per class in the central row and two thousands weights per class in the bottom row. On each column of the table it is possible to see the final mean square error obtained running the algorithm for a different number of iterations.

Observing how the mean square error changes along the rows, it is possible to note that more iterations always lead to a better fit. This is not a surprising conclusion, but as it has been already mentioned, an indiscriminate minimization of the mean square error would lead to plots that have too much a complex shape.

When observing each column of table 4.2 it is also not suprising to observe that increasing the number of investors makes the fit worsen. This is simply due to the fact that the granularity of the model increases, but the number of swaps attempted does not increase accordingly. To put it in simpler words, each perturbation moves a single weight, therefore a swap will modify the allocation in a very marginal way if the total number of weights is extremely high. The only way to achieve an increasing fit without incurring into overfitting is therefore to increase the granularity of the model and the number of iterations simultaneously, as shown by the numbers in the main diagonal of the table 4.2.

This latter conclusion is crucial to understand why the self regularizing search algorithm presented in chapter five has been developed.

Chapter 5

A Self-Regularizing Search Algorithm

This chapter introduces a numerical approach that allows to avoid the overfitting problem observed for the simple search algorithm described in chapter four. This alternative approach can therefore be named self-regularizing search algorithm. The overfitting problem is generated by the simple search algorithm when it is run for an excessive amount of times. The algorithm keeps altering the distribution of weights among different classes of investors, in order to achieve always lower values of the mean square error, but after a certain point the level of complexity of the solution does not reflect any information contained in the data. In other words, aiming to improve the accuracy of the solution the algorithm creates an excessive number of peaks in the allocation of the weights.

It has been observed in the last section of chapter four that there is still a way to improve the fit without incurring into overfitting. When the granularity in terms of number of weights used increases, then a larger amount of iterations can be performed without generating any overfitting problem. It is therefore possible to build an algorithm that simply minimizes the mean square error to the maximum extent possible by letting the granularity of the model increase with the number of iterations performed. This is the main intuition under the self-regularizing search approach, and the main advantage it brings is a simpler setup of the algorithm.

The first section of the chapter presents a detailed description of the approach and highlights all the innovations brought with respect to the simplified algorithm, while the second section shows how the parameter choice affects the outcome and how flexible the approach is when solving the problem for different datasets and different specifications of the agent's utility functions.

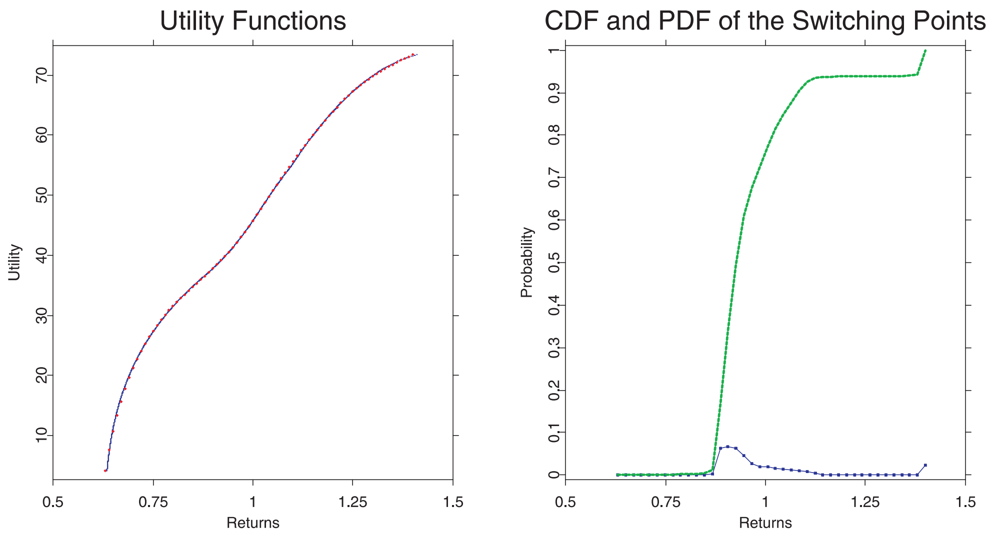


Figure 5.1: Left: Aggregated (Blue) vs estimated (Red) market utility function. Right: estimated allocation (Blue) and cumulative sum (Green) of the weights. Date 30-07-2002

5.1 Description

The self-regularizing approach differs from the simple one for different assumptions on the prior distribution of switching point as well as for the random search procedure.
As in the simple search algorithm, the possible switching points will lie in the range for which the market utility function can be estimated as explained in chapter two. Any number of switching points can be chosen.
Rather than starting from a market where all the possible classes of investors are equally represented, and then perturbating this initial allocation to obtain an allocation that is more suitable to the market data, the self-regularizing approach tries to improve the fit of the aggregate inverse market utility function to the estimated inverse market utility fuction by moving from the representative investor assumption, and then introducing all the different classes of investors with their respective weights.
The prior allocation of weights should replicate the situation of a representative investor with power utility function. In the practical implementation, all the weights are initially put on the switching point that is equivalent to the highest return in the range. Switching in the highest extreme of the range is equivalent to not switching altogether, and putting all the weights in this

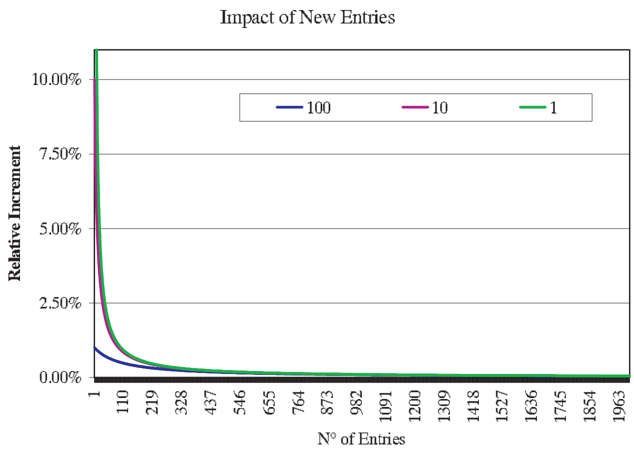


Figure 5.2: Relative impact of new entries under three choices of the granularity parameter

class of investors is equivalent to saying that there is a representative investor with power utility. More precisely the utility function of this representative investor is the low utility level defined in chapter three.
As previously mentioned, the research conducted by Xie (2000) proves that the power utility specification alone is not able to model the financial markets coherently with the empirical evidence. Furthermore it has been shown in chapter two that the nonparametric estimate of the market utility function exhibits a shape that cannot be really approximated using any simple functional specification. Assuming as a starting point that there is a representative investor with power utility function is therefore a wrong assumption, but it will leave room for improvements. These improvements are achieved by allocating new weights to the investor classes available, until it improves the fit. In other words the algorithm moves from the representative investor assumptions and produces a description of a market with many different investors.

The prior allocation Θ^0 is therefore defined by just one parameter, the initial number of weights.
The improvements to the initial allocation Θ^0 are performed using a procedure called entry. This procedure tries to add a new weight to a randomly chosen investor class. The random choice of the investor class is performed using the pseudo-random number generator in the $rand()$ procedure of C++.
It is now worth to mention that the number of investor initially put in the highest switching point affects greatly the iteration that the procedure per-

forms. As it will further described and as displayed in figure 5.2, the choice of the initial number of investors will set the relative importance of the first perturbations attempted.
After the perturbation in the initial allocation of weights is introduced, the procedure computes the mean square error between the market implied return function obtained by aggregation and the estimate obtained from the asset pricing theory. If the fit is improved, the change is kept, otherwise the most recently introduced weight will be eliminated.
An important innovation is that under this approach the algorithm continues to allocate new weights on all the investor classes, until this allows to achieve a better fit. The algorithm stops automatically when more than five thousands random attempts to improve the fit by adding a weight fail.
According to this strategy the number of iterations and the granularity of the model increase in parallel, and as shown in the last section of chapter four this will prevent the algorithm from generating non-smooth solutions. It is therefore possible to state that the self-regularizing approach can be driven only by the minimization of the mean square error without the risk of incurring in overfitting.
Figure 5.1 shows the outcomes of this approach in two plots. In the lefthand side box, the aggregated market utility is plotted as a blue line together with the non parametric estimate of the market utility function obtained from the pricing kernel (red dots). Not only these curves show visually that the fit is very good, but it is should also be noted that the mean square error is 0.000453341, a measure that is not significantly different from zero.
The righthand side box of figure 5.1 shows in blue a plot the estimated optimal allocation of weights $\Theta*$ and in green a plot of the the cumulative sum of the allocated wights. Comparing figures 4.1 from chapter four with the figure 5.1 given above, it can be noted that both approaches deliver almost the same solution. Also with the self-regularizing search algorithm a segment of investors who never switch is observed, represented by the peak in the right tail of the blue curve plotted in 5.1.
In chapter four it has been pointed out that for the simple search algorithm both the choice of the prior distribution and the number of switching points affect the complexity of the solution and allows to improve the fit, provided that a proper number of iterations is to be performed. Due to the lack of any clear methodology to define how the parameters have to be set, the simple search approach presented in chapter four requires a subjective contribution from the user who sets the parameters. The approach presented here sets itself the level of granularity and prevents the overfitting problem from occurring.

Initial Investors	1	10	100
N = 10	0.000627651	0.000486483	0.000514156
N = 40	0.000539867	0.000453341	0.000468398
N = 100	0.000537426	0.000452683	0.000466779

Table 5.1: Mean square error of the estimated allocation using different initial number of investors and number of switching Points (N) parameters. Date 30-07-2002

5.2 Sensitivity Analysis

As described in the previous section the self-regularizing algorithm requires only two parameters to be specified, namely the number of investor classes and the number of weights that have to be used as the starting prior of the algorithm. Although these two parameters have a remarkable impact on the final outcome of the computation of the solutions, it should be noted that the estimated optimal allocation of weights Θ^* obtained using different parameters are now more stable than it has been observed for the simple search algorithm.
The results obtained with this algorithm are pretty satisfactory, and it is therefore interesting to perform stress tests that go beyond the choice of the initial parameters. In chapter three it has been said that power utility functions that describe the low and the high utility level of the individual investors should be fitted to the tails of the market utility function. The tails used up to now reflect high and low utility levels fitted on the returns that belong respectively to the lowest and the highest twenty percentiles of the distribution of DAX returns. This definition of tails can be questionable, and estimates using different tails are presented further on in this section.
In the final part of the chapter it is shown that the self-regularizing search algorithm delivers stable solutions also when modelling the individual utility function of all the investor classes with a logarithmic funcional specification. It is possible to state that the algorithm can compute a solution for all the utility function specifications that are invertible and additive.

5.2.1 Sensitivity to the Prior Allocation

The numerical algorithm described in this chapters requires two parameters to be set in order to start the computation. One parameter is the number of switching point that, as already mentioned in chapter four, is identical

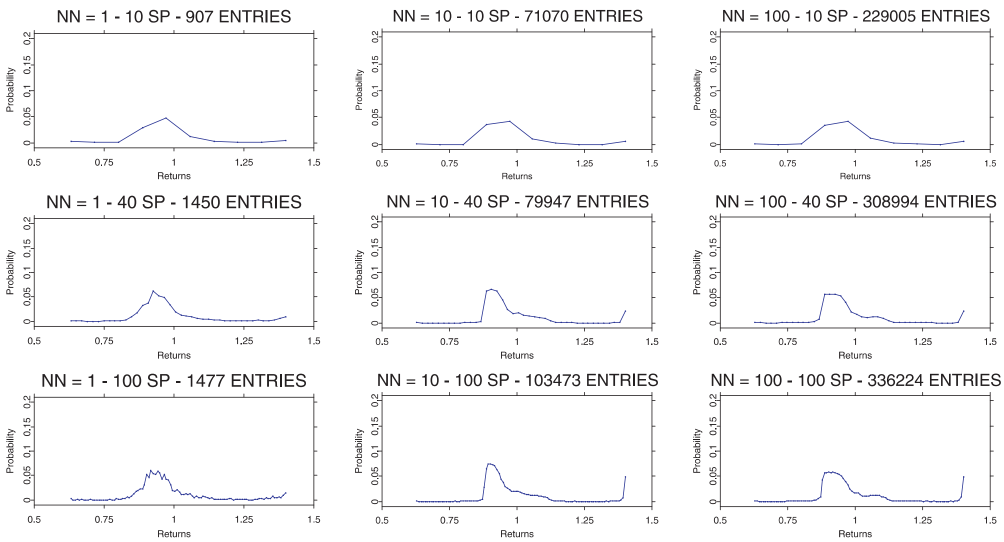


Figure 5.3: Sensitivity of the allocation of weights to the choice of the number of switching points and number of initial investors parameters. Date 30-07-2002

to the number of investor classes that are assumed to exist in the market and is also equivalent to the number of variables that have to be computed by the procedure. The other parameter that has to be set is the number of weights that are initially attributed to the highest switching point in the range. As already mentioned above and displayed in figure 5.2, this parameter defines the relative importance of the first weights that are added to the initial allocation. Figure 5.2 shows three curves for three different values of the number of initial weights parameter, namely a green one obtained starting with one weight, a pink one obtained starting with ten weights and a blue line obtained starting with a hundred weights. It is straightforward to realize that the relative increment brought by an additional weight is much higher for the first entries if the initial number of weights is lower. For an increasing number of additions of weights (or entries) the relative increment of the number of weights generated by an entry converges to zero. Therefore the first additions of weights can be more or less influential, and this will cause the algorithm to reach a situation where no further entry is possible at a faster or slower speed. Conversely when the initial number of weights is very low also the granularity and the accuracy of the solution are low.
Figure 5.3 is a matrix of plots showing the estimated optimal allocation of returns in several setups. The plots contained in the left column of the ma-

trix show the allocations obtained when the initial number of weights is one, while the cental column contains the optimal allocation calculated when the algorithm starts from ten weights, and finally in the column on the right it is possible to observe the optimal allocations computed from a hundred initial investors.

On each line of the matrix are shown the estimated weight allocations computed for a different number of investor classes, and namely ten investor classes in the top row, forty investor classes in the central row and a hundred investor classes in the row on the bottom of the matrix.

Besides the number of initial weights and the number of investor classes, above each plot it is possible to see the number of entries that have been performed. This indicator gives a measure of the granularity of the model, since the discretization of the population of investors into weights is much finer when the number of weights is higher. The number of switching points affects the number of entry modestly. Conversely it is not surprising to observe that the parameter that affects the granularity of the model the most is the initial number of weights. As explained above, starting with many weights means starting with small perturbations.

Observing each column in figure 5.3 it is possible to conclude that when the granularity is very low also the quality of the estimate is poor, and the plot looks not smooth but notched. The plots in the central and in the right column are all smooth and no great differences can be found between the many estimates of the optimal allocation. The plot in the right column tend to develop a sharp edge for future returns around 0.85, and this should discourage from using an excessively high number of initial weights.

The number of switching point chosen is even less influent than it was in the simple search algorithm, and this leaves freedom to the user of the algorithm to determine freely how many classes of investors should be considered.

Once the overfitting problem has been brought under control, if not completely eliminated, it is possible to choose the parameters that minimize the mean square error between the aggregated market implied return and the estimated market implied returns. Table 5.1 shows the mean square error for the different parameter setups. The different mean square errors for the different setups are disposed in table 5.1 according to the same order followed in figure 5.3. Observing each column of the table it is possible to determine that the higher the number of classes of investor considered, the better the fit achieved. On the contrary, it is not always true that for an higher granularity the fit is always better. Since increasing the number of initial weights of the algorithm will make the computation of the optimal allocation extremely long, it is a positive finding to observe that it is not simply by maximizing the granularity of the solution that the best fit can be achieved.

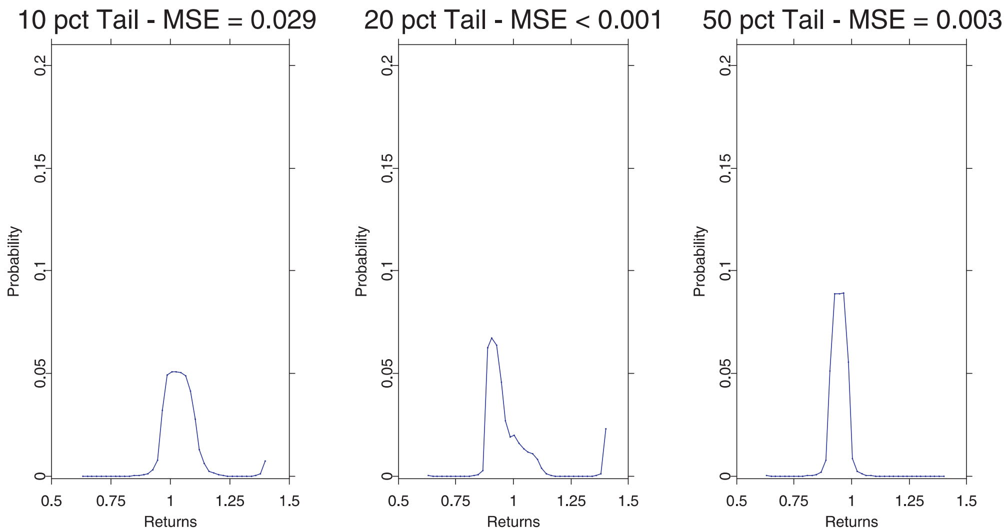


Figure 5.4: Different estimates of the optimal allocation of weights using utility levels fitted on different tails of the market utility. Date 30-07-2002

5.2.2 Choice of the Tails to Fit Low and High Utility

Once the specification of the utility function has been chosen, is then necessary to give a value to the parameters of the utility function. According to the economic interpretation of the implied return functions at both a market and individual level that has been given in chapter three, it is generally stated that the individual utility function for the low and for the high attitude should be fitting respectively to the left and the right tail of the market utility function.

In general it holds that the more extreme the tails are, the better the fitted utility levels will describe the utility of a group of investor that is homogeneously undertaking either the low or the high attitude, since for very low future states almost every investor will perceive utility under a low attitude, whereas for very high future states almost every investor will have switched to an high attitude.

Nevertheless to be able to perform a correct estimate of the utility levels it is also necessary to dispose of a sufficent amount of points, and therefore the tails must not be too small. There is therefore a need to reconcile this trade-off between the quality of the estimated parameters and their economic meaning, and it is worthwile to see how different assumptions about the choice of the tails affect the optimal allocation of weights.

Figure 5.4 is made of three plots, each of them showing the optimal allocation obtained with a different set of parameters for the utility functions. These parameters are already given in table 3.1 in chapter three. The parameters used to obtain the allocation of weights shown in the picture on the left have been fitted on the ten percentiles tails. The allocation in the cental figure is the one obtained with the twenty-percentiles tails and has been used for all the analysis or the simple search and self-regularizing search algorithms presented in chapter four and two. Finally, the optimal allocation of weights presented in the plot on the right shows the optimal allocation of weights obtained when the two tails correspond to utility levels fitted on the fifty-percentile tails. Above each plot the mean square error computed for each estimated optimal allocation is given.
A remarkable feature is that in the latter case, as to say when the estimate of the utility levels is done using the full range of return considered, it is possible to observe that the distribution of the switching point doesn't present the peak on the right end that has been observed in figure 5.1 and that is visible in the central plot of figure 5.4. When using the power utility function specification, it is possible to observe a trade off between the choice of extreme tails to fit the utility levels, and the peak on the right end of the estimated allocation of weights. This peak is in fact not specific of the estimate run using the data of day 30-07-2002, but as will be shown in the next chapter all the estimates in 2000, 2002 and 2004 allocate from 2% to 3% of the total weights on the last switching point.
However, as shown in the plots of figure 5.4 the choice of the tails does not affect the meaning of the final outcome significantly. For larger tails the solution is more concentrated to the left, as to say investors are supposed to switch in a smaller range of returns. As shown in the following section, the impact of the choice of the tails is different according to the utility function considered.

5.2.3 Choice of the Utility Specification

The algorithm can be succesfully applied also when the utility function is differently specified. To prove the flexibility of the approach it could be possible to choose a straightforward log-utility specification as described in the equations 5.1 and 5.2.

$$U_L^L = a_L \cdot log\,(X_T) + b_L \tag{5.1}$$

$$U_H^L = a_H \cdot log\,(X_T - shift_i) + b_H \tag{5.2}$$

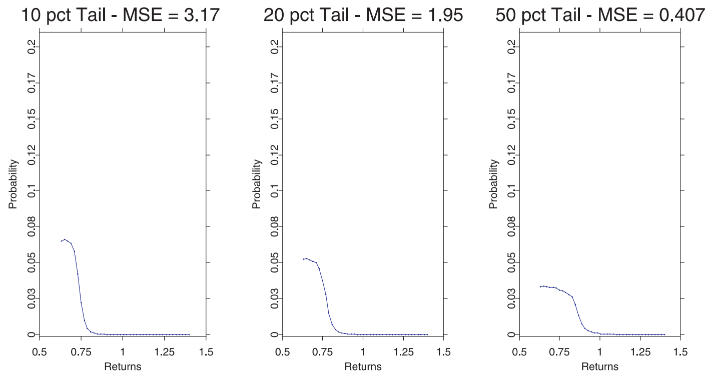


Figure 5.5: Different estimates of the optimal allocation of weights using logarithmic specification. The utility levels are fitted on different tails of the market utility. Date 30-07-2002

The utility functions 5.1 and 5.2 can easily be inverted and used for the aggregation process has it has been explained in chapter three for the power utility case. Thus, several parameter estimate can be performed according to the choice of different tails, as well as using market data observed on different dates.
The estimates assuming that agent's preferences are described by logarithmic functions as presented in figure 5.5. Figure 5.5 containes three boxes, each of them is showing a different estimate of the optimal allocation of weights among investor classes, according to different estimates of the parameters of the high and low utility levels. In the plot on the left, the parameters for the low and the high attitude have been fitted to the tails of the market utility functions that are representative of the lower and higher ten percentiles of the DAX returns observed in 2002. The box in the center contains the optimal allocation obtained when the utility levels are fitted to the twenty-percentiles tails. Finally, the box on the right presents the estimated optimal allocation of weights among investment classes that is computed when the tails are defined on the higher and lower fifty percentiles of the DAX returns. The self-regularizing search algorithm delivers smooth plots even for the logarithmic utility function specifications, and is performing well also for different choices of the tails. Nevertheless in comparison with the power utility function specification, the fit obtained performed an aggregation of investors that follow logarithmic functions is very poor, as the MSE reported above each plot of figure 5.5 shows.
The choice of the tails to be used for fitting the high and the low utility levels has a more pronounced effect here rather than observed for the case of the

power utility function. In this case there is no doubt that different assumptions on the tails allow a better fit of the aggregated inverse market utility function to the estimated inverse market utility function obtained according to the approach described in chapter two. Furthermore, in the logarithmic case the choice of the tails distorts the final estimate of the allocation more than in the power utility case. While in the former case the larger were the tails the more concentrated was the estimated optimal allocation of weights, comparing the plot on the left with the plot on the right of figure 5.5 an opposite behaviour is observed. Under the assumption of logarithmic individual utility functions, a choice of more extreme tails lead to a more concentrated optimal allocation of the weights among classes of investors, while when also more central observations are taken into account then the distribution of the switching points is more dispersed.
The experiment presented in this section shows that the results obtained under different assumptions on utility specifications are each of them a story on its own. It is indeed not possible to make any general conclusion about the way the choice of tails affects the final result. In turn the finall estimated allocation looks remarkably different when changing the individual utility specification, as a quick comparison between figure 5.5 and figure 5.4 proofs. Although the logarithmic utility specification doesn't allow to reconcile individual utility with the market quotes, this experiment has shown that the algorithm itself is flexible enough to deal with different setups of the problem.

Chapter 6

Application of the Algorithm

In the previous chapters the theoretical framework assembled by Moro, Detlefsen, and Härdle (2007) has been presented as well as ways to solve the aggregation problem stated in chapter three. It is finally possible to describe the insights in capital markets that are made possible by the aggregation technique and give a more precise understanding of the behaviour of the investors in the market.
The theoretical framework presented in the present work allows to decompose the market into different profiles of investors, simply disposing of DAX quotes and ODAX quotes. Not only it is possible to understand what kind of returns are perceived as low and which as high by the market but also how. Assuming that the aggregation procedure is a reliable tool to profile investors from option data, it is also possible to test different assumptions about the utility function of the individual participants in the market, and verify wether they provide results that are consistent with the market data observed.
In the first section of the chapter the above mentioned insights in capital markets are introduced, while the second section uses the aggregation approach as a mean to test the simpler description of investor preferences based on logarithmic utility and eventually discard it.

6.1 Describe The Attitude of Investors

A good estimate of the optimal allocation of weights allows to represent in a comprehensive model the microeconomic behaviour of the agent and the macroeconomic behviour of the market.
The assumption is that the market can be described by the representative agent model introduced in chapter two, while the investors at the microe-

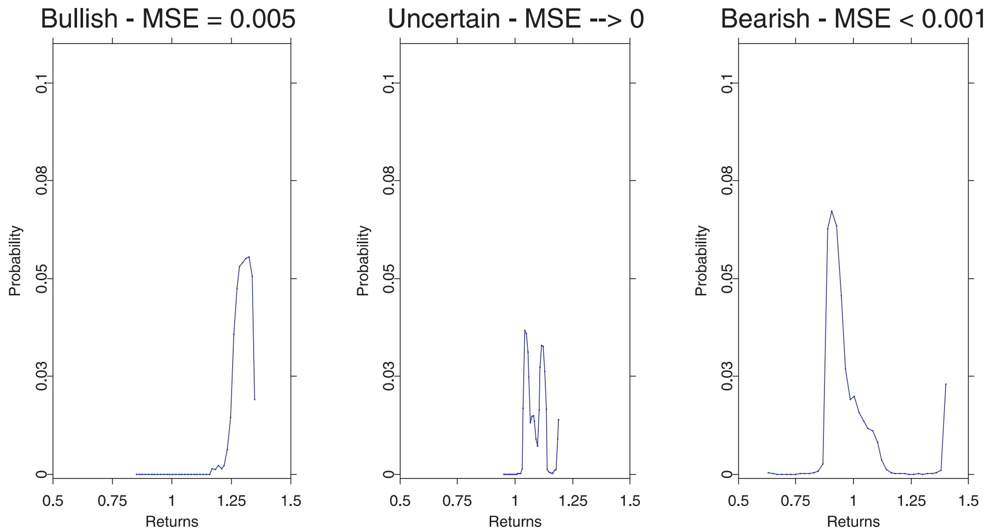
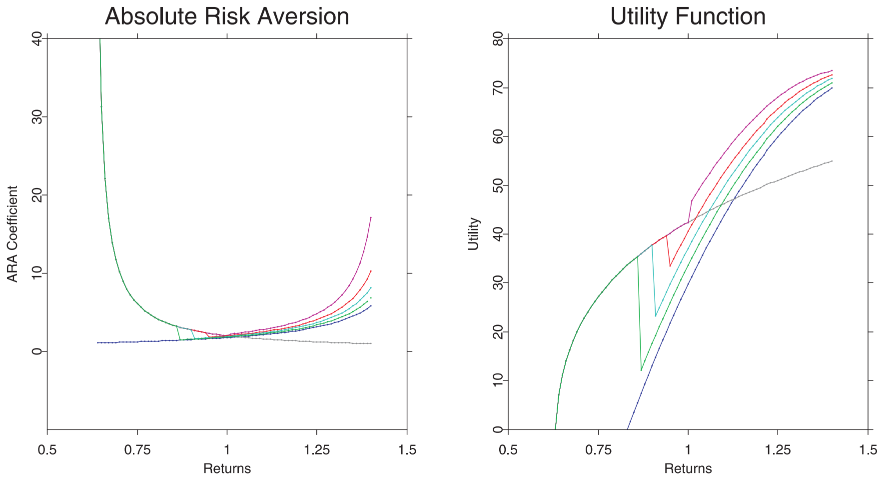


Figure 6.1: Estimated optimal allocation of weights on 24-03-2000 (left), 30-06-2004 (center) and 30-07-2002 (right)

conomic level are described by the kinked power utility functions that are presented in chapter three.
It is furthermore assumed that there are several types of investors, each of them identified by a switching point. For each class of investors, the switching point is the interception between the utility function under the low attitude and the utility function under the high attitude. This description allows to describe how each class of investors is affected by a distortive effect when perceiving the utility achieved in case of higher returns, rather than for lower returns.
The main reasons to assume the existence of distorsive effects that affect the perception of individuals are three. First of all it is logical to believe that each investor is different due to his or her specific endowment, expectations, and any other rational or irrational element that affects his or her position about future returns of stock markets. Furthermore the empirically documented phenomenon called reflection of preference (see Kahneman and Tversky (1979)) shows that investors prefer a certain gain than an uncertain one with equivalent expected value, while a certain loss is seen as inferior with respect to an uncertain loss with the equivalent expected value. Finally, the third reason lies in the irregular shape of the estimated market utility function presented in chapter two in figure 2.3. This shape can hardly be described by a functional specification, but can be well approximated by

30-07-2002



30-06-2004

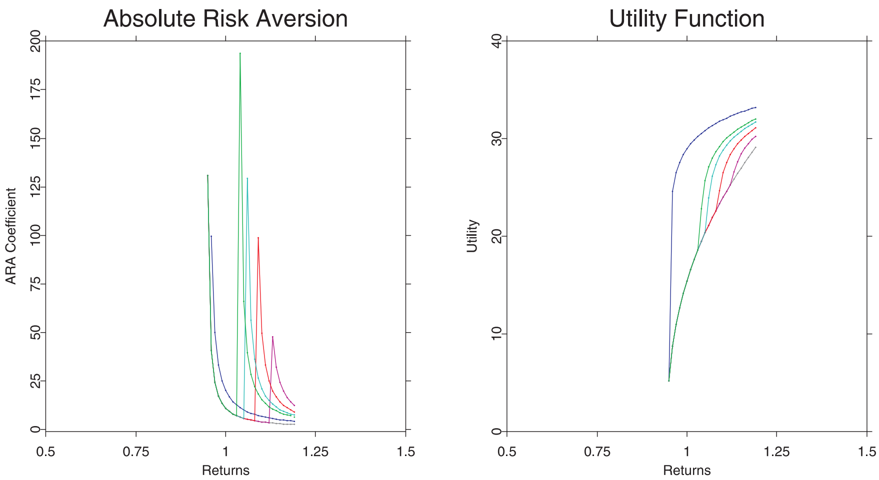


Figure 6.2: Utility and absolute risk aversion of investors switching at the percentile 0 (blue), 1 (green), 25 (light blue), 50 (red), 75 (pink), 100 (grey) of the distribution of weights estimated using data from 30-07-2002 (above) and 30-06-2004 (below).

two simpler curves. While the first argument justifies the existence of several classes of investors, the former two directly support the representation based on pairs of high and low utility levels presented here.
The numerical procedure introduced in chapter four and five, finally allows to perform the aggregation of the classes of investors into a market. Assuming that the estimate for the market utility performed in chapter two is correct, and assuming that the aggregation procedure is able to bring together the investors types, it is eventually possible to describe how many investors from each type are present in the market and how they behave. All the estimates presented in this chapter follow the self-regularizing approach, set the initial number of weights equal to ten and the number of switching point to forty. For the financial data collected on 30-07-2002, these results are presented in chapter five.
Especially observing the right box of figure 5.1 it is possible to gain a valuable insight in the perception of the future states that the market show. As already mentioned, the blue line in that plot is showing the optimal allocation of the weights among several classes, while the green line is the cumulative sum of the weights. Looking at these two lines plotted in figure 5.1 is therefore equivalent to looking at the probability density function and at the cumulative density function of the switching points.
A first thing that strikes the attention is that a significant amount of investors perceive as high even absolute returns that are actually below one, and are therefore losses. In other words, most individuals perceive future states of consumption according to the high utility level even when their investments in capital markets incur into losses. Conversely, other individuals remain on the low levels of utility until they achieve positive returns. Finally a third segment of investors are those who never switch, and are located in the last switching point available. Although this feature of the plot might be just a product of the assumptions taken on the utility specification and on the choice of the tails, it is not strange to observe a segment of investors that are making choices according to the power utility function specification.
It is not very easy to give a concrete meaning to the switching points and the high and the low attitudes, since these are an invention that helps to describe preferences rather than an observed feature of the human beings. Nevertheless the concept of the reflection of preferences mentioned above helps to read the results of figure 5.1 in an applied way. While in economic theory it is generally assumed that investors are risk averse, the main finding related to the reflection of preferences is that when states are bad, many individuals want to undertake risk and gain at least a chance to achieve better results rather than sticking to a certain loss. In the model considered here, investors are always risk averse, nevertheless, as it will be introduced later and shown

in figure 6.2, the investors are less risk averse when they undertake the low attitude.
Since choices are made according to the unexplicable mechanisms of the human brain, it is hard to say precisely what motivates such an attitude. Still, several hypotheses can be done. A first possible explanation is that some investors have a limited exposure to downside risk. A possible example is when an investor is already expecting to lose a very big part of his or her investment, but the losses cannot be bigger than the amount invested. Although on one side an investor in such a situation will want to protect the safe part of his or her capital, it is clear that more risk is mostly an opportunity. Another example is given by a consumer who can't fully reach his or her level of sustenance and can't be satisfied with his or her income. For the latter kind of agent the expected state is so bad that an opportunity to improve it will be much more important than the risk of worsening it. The two examples made above might be a bit too extreme since no risk seeking agents are observed in the market. Nevertheless they help to clarify why should an agent be less risk averse when looking at bad future states rather than when looking at good future states. As it will be explained below in this section, this hint allows for the existence of a category of investors that always undertake the low attitude, in the case the peak observed on the right tail of the switching point density of 5.1 is not a product of the numerical procedure.
A better analysis of the results obtained in chapter five can be performed looking at figure 6.2. Once the switching points have been chosen and once the utility levels have been defined, the set of functional specifications for the utility functions is finally available and it is possible to derive the risk aversion according to the Arrow-Prat definition given in equation 6.1.

$$ARA(x) = -\frac{U''(x)}{U'(x)} \tag{6.1}$$

Figure 6.2 is made of four plots. The row above shows the estimates performed fitting the coefficients of the high and low attitudes to the market utility function estimated on the 30-07-2002, while the row below presents the estimates using coefficient fitted on the estimated market utility on 30-06-2004. In each column, for specifically significant type of investors, the lefthand box contains plots of the absolute risk aversion, whereas the righthand plot shows the total utilities.
The several curves plotted in the four boxes of figure 6.2 show the utility and the risk aversion functions of five types of agents, that in turn are representative of a specific quantile of the estimated optimal allocatoin of weights among classes. The first curve (blue) identifies the type of investors that

always follow the high attitude. It is representative of almost no investors altogether, it is however interesting to note how this type adopts a more risk averse attitude even for very low returns. On the contrary, the grey curve represents the risk aversion function of the group of investors that never switch. By comparing these two group it is possible to see how different the two risk aversion functions are shaped. The risk aversion of the low attitude is always lower than for the high attitude and it is decreasing over returns. As expected, the high utility level presents generally higher risk aversion. Generally, the absolute risk aversion should be decreasing over higher returns, whereas the plot for date 30-07-2002 shows that this is not always the case. The plot on the top-left position of figure 6.2 shows indeed that for the high attitude the risk aversion is increasing over returns, while the plot in the bottom-left position contains a set of risk aversion functions that look like what has been expected, and are decreasing for higher returns. The origin of this difference is to be found in the coefficients estimated for the low and for the high utility level summarized in table 3.1 of chapter three. The nonparametric market utility function in these two dates has a remarkably different shape, and the best approximation concerning the 2002 case has a d_2 coefficient equal to two, whereas in 2000 and in 2004 the coefficients are respectively 0.25 and 0.0028. The exponent of the power utility function is the most important parameter for the computation of risk aversion, since it is present in both the first and second order derivative. It can be observed that for d_2 parameters in the $[0, 1]$ range the risk aversion is a decreasing function. Approximations of the right tail of the estimated market utility function in 2002 have been tried using other parameter sets, but the quality of the results obtained has been inferior than what has been presented here. It is therefore preferred to tolerate this exception in the shape of the risk aversion function.

Observing the sequence of curves in each plot from the left to the right, four coloured curves represent the investor classes that switch respectively in the first, twenty-fifth, fiftieth are seventy-fifth quantiles. For year 2002, these curves clearly show that the biggest cluster of investor started considering high the future states that are in the range of returns between 0.85 and 1. Roughly 25% of the market perceives as high only positive returns.

It shall be noted that the utility functions are not time continuous, and for each date a different estimate of the parameters has to be performed. Therefore also the estimate of the optimal allocation of weights will be very different according to the day in which the market data has been collected. For year 2004 it is shown in the bottom plots of figure 6.2 that the switching points are mostly found between the 1.1 and the 1.20 absolute returns.

Figure 6.1 shows the estimated allocation of weights over the investor classes

in the three different dates already mentioned in chapter two. The first plot on the left presents the estimate performed for a bullish market momentum in year 2000. The plot in the center presents the allocation of the weights that has been estimated for year 2004, when the market didn't show any clear trend. Finally in the plot on the right the estimate performed in the bearish markets in 2002 is given.
The most striking feature that can be observed is that the more the market performs well, the higher will be the future state in order to be perceived as high.
The plot on the left of figure 6.1 shows that when the markets had a bullish trend all investors required positive returns to switch to the high perception of future states. In this case investors are clearly more ambitious, and they turn to the high attitude only for returns that are in the [+25%, +40%] range. As previously noted investors start to switch to the high attitude even when the returns are negative when the markets are bearish.
Furthermore under the uncertain market conditions in 2004, that don't show either an extremely positive or an extremely negative trend, two peaks can be observed as shown in the central plot of figure 6.1. A first class of investors has moderate expectations and will be satisfied even with relatively small returns, similarly to what has been observed for the case of the bearish markets in 2002. A second cluster consists of agents that will perceive as high only higher returns, similarly as what observed for the bullish market of 2000.
Again, on the extreme right of the plot, a number of investors who actually never switch are allocated, as in the case of bearish markets. As already anticipated in chapter five, a similar percentage of investors that never switch is found for every day considered by our analysis.
It is especially interesting to look at the plots in figure 6.1 and the above described conclusions considering that the six months historical average return on date 24-03-2000 was +24%, whereas on date 30-07-2002 it was -2% and on date 30-06-2004 was +12%. The six months historical average return is defined as the average of the performance of the DAX index over six month in a time window that goes six months back from the reference day. This kind of historical return is particularly interesting because, how stated in chapter two, the option quotes used to estimate the market utility function have a six months time to maturity. It is therefore significant to notice that the weights are mostly allocated in the switching points that are close to the six month historical returns, as if this historical indicator is one element that the individual investors as modelled using the approach of Moro, Detlefsen, and Härdle (2007) take into account when building their expectation about future states.

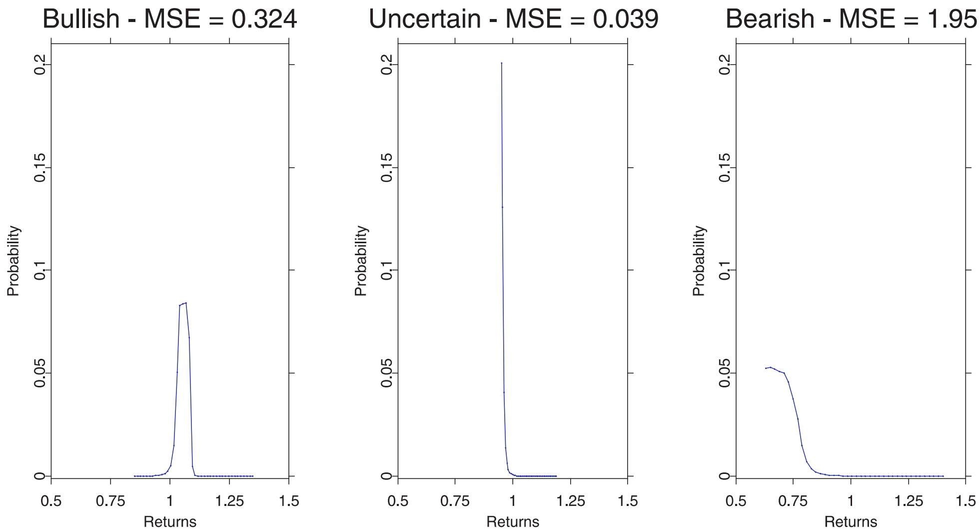


Figure 6.3: Estimate of the optimal allocation of weights on 24-03-2000 (left), 30-06-2004 (center) and 30-07-2002 (left) based on logarithmic utilities

This element in support of the quality of the representation of individual preferences is indeed very important, since if the representation of the individual preferences introduced in chapter three can be considered reliable, the allocation of weights obtained from the aggregation approach can be used to describe the expectations about capital markets that investors have, their profiles, their outlooks and their strategies. An asset manager could therefore perform the analysis presented here using current data on options with six or twelve month maturities, and quickly obtain a representation of the expectation of the market participants. Using this data, the asset manager can improve his or her middle and long term investment strategy, and develop products that will satisfy the needs of the different investor profiles obtained under this approach.

6.2 Test Another Utility Specification

In the previous section it has been shown how the aggregation procedure based on two utility levels and the power utility specification can produce a good description of classes of investors with different profiles. Another possible application of the aggregation approach has a more academic interest.

Using the aggregation procedure to build a bridge between market returns and individual preferences, it could be possible to use this approach to test an utility specification that might be considered suitable to model the preferences of individual investors.

As already mentioned in chapter five, the numerical procedures developed for this work are flexible and can be used with several functional specifications of utility. The results obtained using the log utility defined by equations 5.1 and 5.2 are presented in the last section of chapter five by the figure 5.5.

Also for the logarithmic utility the switching points have been set to be forty and the initial number of investors is ten. Finally the utility level are fitted on the twenty-percentiles tails. The estimates are presented in the three boxes of figure 6.3 using market quotes for the dates 24-03-2000, 30-07-2002 and 30-06-2004. These three estimates are plotted respectively in the left, central one, and right boxes.

At a first glance, the three plots demonstrate that the logarithmic utility simply doesn't provide a realistic description of the individual preferences. For the bullish market as observed in 2000 the plot doesn't show in principle any unpleasant feature, although the switch to the high attitude does not happen anymore for the very high returns as observed in figure 6.1. The switching points are generally lower for all dates, and this feature causes some problems to arise for the two scenarios where the switching points are lower. The optimal allocation computed in the 2000 and 2004 cases is indeed too concentrated on the left margin of the return range, and this suggests that a description of the individual preferences based on the log utility leads to excessively low switching points. In the case of the uncertain and the bearish market, the distribution of switching points seems to require a different interval of returns than the one that is actually realistic to consider, while in the power utility case the largest part of the switching points fall within the range taken into account.

Another point against the logaritmic utility function is the fit between the aggregated and the estimated inverse market utility. As shown by the mean square error indicator above each plot of figure 6.3, the fit is always poorer than in the power utility case.

Chapter 7

Conclusions

The aggregation approach implemented in the macroeconomic model of Moro, Detlefsen, and Härdle (2007) provides a framework to bring together the preferences of groups of investors with the information contained in financial markets quotes. This approach, introduced in chapter two, is build from a wide range of contributions from economics and statistic. Across all the previous chapter it has been described how to use index quotes and the quotes of its relative option contracts to derive an estimate of the market utility function, and how to reconcile assumptions about individual investors' behaviour with the estimated utility of the market. The heart of the approach from Moro, Detlefsen, and Härdle (2007) is that, once a model for the behaviour of classes of investors has been set up, it is then possible to quantify the size of each investor profile by solving the aggregation problem presented in chapter three.

From a mathematical perspective the aggregation approach is an ill posed optimization problem, whose solution are the weights that are associated to each investor profile. Rather than being analitically derived, the weights can be obtained using a random search process that finds the best fit of the preference set of the aggregated market to the preference set derived from the market data.

Two numerical approaches are implemented, moving from different assumptions and following two different procedures to search the solution. As a first result it shall be noted that both approaches lead to the identification of a similar pattern in the allocation of weights among investor classes. The description of the market achieved with the aggregation algorithm is also coherent with both the economic model that has been developed and the market conditions in which the data has been collected. The implemented approaches are therefore successful in determining a good representation of the investor profiles from market data.

The first algorithm developed requires the subjective choice of several parameters, and this represents an obstacle to the correct identification of a smooth estimate of the allocation of the investors. Once the number of switching points and the interval they belong to are chosen, a peculiar limitation arises from the combined action of these two parameters with the number of iterations that the algorithm performs before stopping. This latter parameter is used to regulate both accuracy and smoothness and a wrong choice might lead to overfitting. A good choice for the number of iterations depends upon the level of granularity that has been set to discretize the market into unitary weights. The granularity is determined by the number of weights, that in turn is equivalent to the product of switching points and the number of weights per switching point. Furthermore, the higher the number of weights, the higher will be the number of swaps that have to be performed to achieve a good estimate. Therefore the number of iterations must be corrected according to different assumption on the number of switching points and on the number of weights.

The second algorithm implemented is based on a search procedure that adds weights to the prior until the allocation is improved to the minimum level of mean square error achievable for the given parameter set. Besides the number of switching points and the range of returns where they must lie, the algorithm further requires only a to define how many investor are present in the market. The latter parameters affect the granularity of the final solution and the number of iterations is found accordingly. The number of investor classes and the initial number of weights parameters don't really affect the informative content of outcome of the procedure, whereas they are useful to balance smoothness and accuracy. In particular the initial number of weights adds or subtracts importance to the first additions done to the prior allocation. Under this approach it is possible to determine the best parameter simply by minimizing the error of fit without incurring into overfitting.

Profiling market investors opens new opportunities for asset managers as well as any retailer of financial products who seeks to adjust his or her product range to better match the needs expressed by the market. According to a scientific procedure it is indeed possible to define the preferences of investor classes and then estimate the relative size each investor class with respect to the market.

Potentially the aggregation procedure is able to give to asset managers and risk manager the possibility to conduct a survey of the market sentiment in a time span of few minutes.

It has indeed been observed that starting from an artificial classification of individual investors inspired by the research of Kahneman and Tversky (1979) and Moro, Detlefsen, and Härdle (2007), the aggregation approach allows to allocate weights to all the investor classes defined. The way all the investor classes are represented in the market is reconciled very well with the information reflected in the market quotes of the option contracts traded in the German markets.

According to the way investors have been defined, it is possible to state for each investor class a definition of low and high returns. It is actually very encouraging to note that, according to the assumptions about investor's behaviour presented in this work, the aggregation approach brings the instance of individuals together with the market situation in a consistent way. It has been demonstrated that when the markets are experiencing a bubble all the investor shift their definition of high return up. Viceversa in the bearish phase that followed the 9-11 event, all the investors perceived even negative returns as high.

The individual preferences are described in this work only according to evidence from literature, but the numerical approaches introduced here offers the flexibility to bring together a wide range of utility specification for the individuals preferences with the market utility estimate. Once more information about the individual preferences is made available, it will also be possible to adjust the microeconomic framework according to these new findings, and obtain even more interesting results.

Although the kinked power utility framework adopted here is already performing very well, the description of individual preferences could also be further improved. A possible approach would be to perform surveys among investors and try to implement a nonparametric representation of the individual's utility, in order to gain maximum flexibility.

In order to achieve a theoretically sounder framework, improvements could be made also on the aggregation mechanism itself.

Rather than the average of the implied returns, the market price per state could be obtained through a trading simulation. This different approach has already been presented by Sharpe (2006) and can be reconciled with the capital asset pricing model, a common tool used by several funds.

The trading simulation is promising but it is still a green concept that will still require a significant amount of new research. No trading mechanism has at the present time been developed and tested empirically, and even the theoretical framework presented by Sharpe (2006) has not yet been proven to be reliable. Although more realistic in principle, when performing the aggregation via trading simulation issues about complexity and transparency of the computations arise. It is indeed necessary to perform assumptions about how the market participants perceive utility and how the trade is performed, and it is to be expected that the quality of the final result will depend strongly on the quality of the assumptions about the trading mechanism. Therefore

at the state of the art even the trading simulation will only deliver an approximation of the market dynamics, exactly as the aggregation approach presented here does.
Furthermore, performing estimates of the market utility functions on a larger time window will allow to further test the effectiveness of this approach using the statistical tools made available by econometrics.
The macroeconomic model that describes the market as a representative agent presented here is a very simple although capable one. Interesting improvements could be done, for instance taking in consideration several types of financial market and several financial products.
Considering the above mentioned margins of improvement, the model developed by Moro, Detlefsen, and Härdle (2007) that has been implemented in this paper is already able to provide a simplified representation of the market as well as the individual preferences. Until new findings will open the way to more sophisticated aggregation procedures, the approach presented here can still be considered a competitive tool. Although several concessions to simplicity have been done, extremely unrealistic assumptions have been avoided, and the results obtained are encouraging.
For each category of investors that has been defined, it is now possible to derive an utility function that can be applied in the context of a Markowitz return-utility investment choice. The weights allocation obtained from the aggregation problem reflects the informative content of market quotes. This deeper insight in the investor profiles will allow the asset manager to develop strategies and products that will be better matching the preference and the perception of all the investor classes. Thus, disposing of an understanding about the importance of each investor class, the asset manager will be able to identify interesting market niches, and therefore gain in terms of competitiveness.

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